Fluid Mechanics

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1 FLUID & ITS PROPERTIES

1.1 Introduction

1.1.1 Fluid

Fluid is the phase of substance which can't resist any external shear force in static condition.
Fluid ⇒ Continuous deformation under the action of shear force.

$$\tau \propto \frac{d\theta}{dt}$$

Here $\frac{d\theta}{dt}$ is deformation rate (or) shear strain rate during steady state.

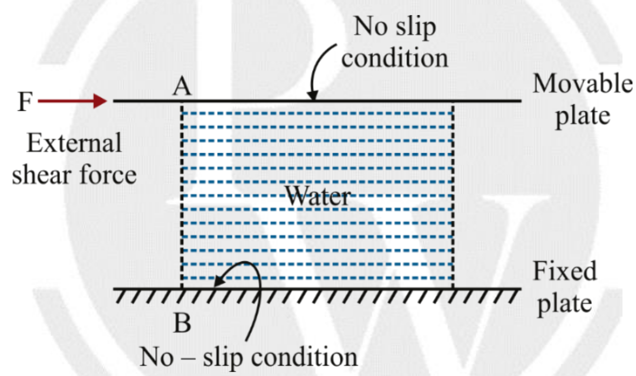


Fig. 1.1 Fluid between two parallel plates (Fixed and Movable)

1.1.2 Fluid Properties

Mass Density / Density:

Density is defined as the mass per unit volume.

Mathematically

Density $(\rho) = \frac{\text{mass (m)}}{\text{Volume(V)}}$

On Increasing Temperature ⇒ Density decreases.

On Increasing Pressure ⇒ Density increases.

Units:

(i) kg/m^3,

(ii) gm/cm^3

> Note:
> $$1\frac{gm}{cm^3} = 10^3\frac{kg}{m^3}$$

Dimensional formula: $[\rho] = [ML^{-3}]$

Specific volume:

Specific volume is defined as the volume per unit mass.

Mathematically

$$\text{Specific volume, } (v) = \frac{\text{Volume }(V)}{\text{mass}(m)} = \frac{1}{\rho}$$

Units:

(i) $\frac{m^3}{kg}$,

Dimensional formula: = $[M^{-1} L^3]$

Weight Density/Specific Weight (w):

Weight density / Specific weight is defined as the weight per unit volume.

Mathematically,

$$\text{Specific weight } (w) = \frac{\text{Weight}}{\text{Volume}}$$

$$w = \rho g$$

On increasing Temperature ⇒ Specific weight decreases.

On increasing Pressure ⇒ Specific weight increases.

Units:

(i) $\frac{N}{m^3}$, (ii) $\frac{kg}{m^2-s^2}$, (iii) $\frac{gm}{cm^2-s^2}$

Dimensional formula: $[w] = [ML^{-2}\ T^{-2}]$

Specific Gravity (s):

Specific gravity is defined as the ratio of density of fluid to the density of standard fluid.

$$\text{Specific gravity (s)} = \frac{\text{Density of fluid }(\rho)}{\text{Density of Standard Fluid }(\rho_s)}$$

$$S = \frac{\rho}{\rho_{water}}$$

- Water is taken as standard fluid.
- Water is Specific gravity is **UNITLESS**.

Note:

Relative density term is used instead of specific gravity if comparison is done with respect to some random fluid instead of standard fluid.

1.2 Viscosity

1.2.1 Viscosity of Liquid (μ)

When the liquid is in motion, the internal resistance offered by one layer of liquid to the adjacent layer of liquid is known as viscosity.

- In any fluid viscosity is because of
  (a) Cohesive force between fluid particles.
  (b) Molecular momentum exchange
- For liquid at rest or relative rest, the concept of viscosity is not valid
- For Ideal fluid, cohesion is zero hence viscosity is also zero.

Newton's Law of Viscosity

$$\tau \propto \frac{d\theta}{dt}$$

$$\tau = \mu \frac{du}{dy}$$

$$\Rightarrow \qquad \mu = \frac{\tau}{du/dy}$$

Here,

μ =Cofficient of viscosity or absolute viscosity or dynamic viscosity

Units:

(i) $\frac{N-s}{m^2}$ (or)$Pa - s$, (ii) $\frac{kg}{m-s}$ (iii) $\frac{gm}{cm-s}$ (poise)

Dimensional formula: $[\mu] = [ML^{-1}T^{-1}]$

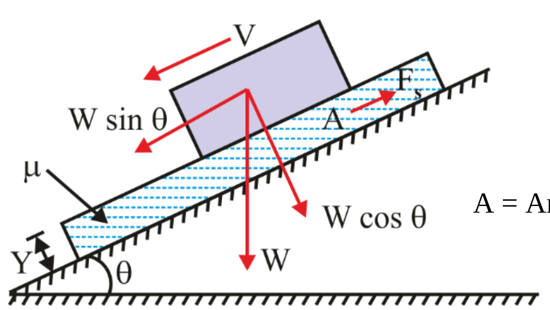
Note:

(i) 1 poise = 10^{-1} Pa-s

(ii) 1 CP = 10^{-2} Poise

1. 2.2 Different Cases of Viscosity

Block Sliding Downward on Inclined Surface Due to Self-Weight:



A = Area of contact between block and fluid

Fig. 1.2 Fluid between two parallel inclined plates (Fixed and movable)

Note:

(i) At $\theta = 0°$, then $V = 0$ (Horizontal surface)

(ii) At $\theta = 90°$, then $V = \frac{WY}{\mu A}$ (Maximum terminal velocity for a given W, Y, μ and A)

At equilibrium

$$F_s = W \sin\theta$$

$$\tau A = W \sin\theta$$

$$\mu \times \frac{V}{Y} A = W\sin\theta$$

$$\Rightarrow \quad V = \frac{(W \sin\theta)Y}{\mu A}$$

Thin Plate Moving in Horizontal Gap:

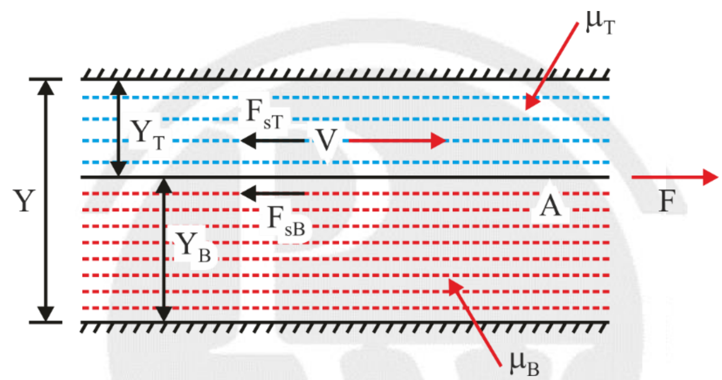


Fig. 1.3 Thin plate moving in horizontal gap

Here,

A → surface area of plate

V → velocity of plate

At equilibrium

$$F = F_{sT} + F_{sB}$$

$$F = \left(\frac{\mu_T}{Y_T} + \frac{\mu_B}{Y_B}\right) VA$$

Special Case:

- For same shear force on both sides $\frac{Y_T}{Y_B} = \frac{\mu_T}{\mu_B}$
- For minimum pulling force $\frac{Y_T}{Y_B} = \sqrt{\frac{\mu_T}{\mu_B}}$

Solid Cylinder Rotating in Hollow Cylindrical Cavity

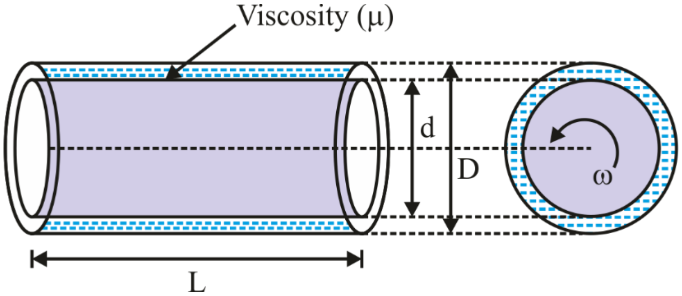


Fig. 1.4 Fluid between two concentric cylinders (One rotating and another fixed)

- Shear stress $\tau = \frac{\mu\omega d}{2Y}$ $\left(Y = \frac{D-d}{2}\right)$
- Shear force applied to rotate solid cylinder $F_s = \frac{\pi\mu\omega d^2 L}{2Y}$
- Torque Applied $T = F_s \times r = \frac{\pi\mu\omega d^3 L}{4Y}$
- Power Required to rotate the solid cylinder $P = T \times \omega = F_s \times V = \frac{\pi\mu\omega^2 d^3 L}{4Y}$

Circular Disk:

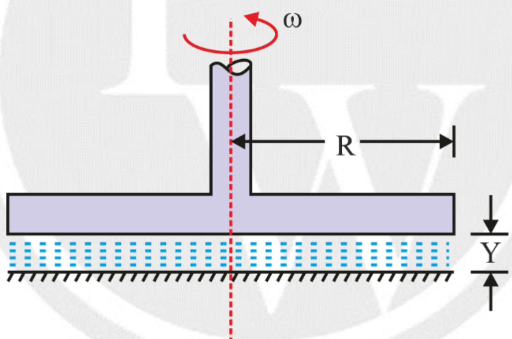


Fig. 1.5 Fluid between two circular disks (One fixed and another rotating)

- Shear force applied to rotate circular disc $F_s = \int_0^R \tau dA = \int_0^R \mu \frac{r\omega}{Y} \times 2\pi r dr = \frac{2\pi\mu\omega R^3}{3Y}$
- Torque Applied $T = \int_0^R \tau dA \times r = \frac{\pi\mu\omega R^4}{2Y}$
- Power Required to rotate to circular disc $P = T \times \omega = F_s \times V = \frac{\pi\mu\omega^2 R^4}{2Y}$

Solid Cone:

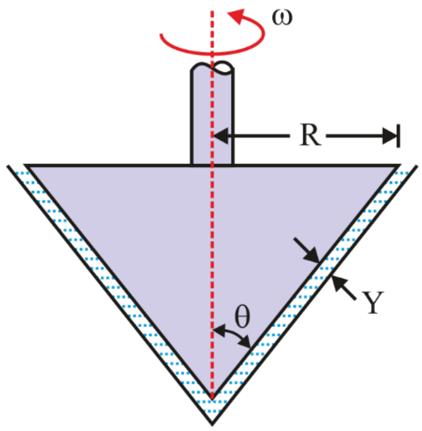


Fig. 1.6 Fluid between a hollow and solid cone

- Shear force applied to rotate solid cone $F_s = \int_0^R \tau dA = \frac{2\pi\mu\omega R^3}{3Y} \times \frac{1}{\sin\theta}$
- Torque Applied $T = \int_0^R \tau dA \times r = \frac{\pi\mu\omega R^4}{2Y} \times \frac{1}{\sin\theta}$
- Power Required to rotate to solid cone $P = T \times \omega = F_s \times V = \frac{\pi\mu\omega^2 R^4}{2Y} \times \frac{1}{\sin\theta}$

Note:
1. For liquid, on increasing temperature cohesion decreases leading to decrease in viscosity
2. For gases, on increasing temperature further increases molecular collisions, so viscosity increases

1. 3 Kinematic Viscosity (v)

- It is ratio of dynamic viscosity of fluid to density of fluid.

$$v = \frac{\mu}{\rho}$$

- $\nu \propto T^{3/2}$ (for ideal gas) and $\mu \propto \sqrt{T}$ (in general) where T = Absolute temperature.

Units: m^2/s

Dimensional formula:

$$[v] = [L^2T^{-1}]$$

Note:
1 stoke $=10^{-4}$ m^2/s = $1 cm^2/s$

1. 4 Compressibility & Bulk Modulus

1.4.1 Compressibility (β_c)

Compressibility of a fluid refers to its ability to change its volume and density when subjected to pressure. The coefficient of compressibility β_c is defined as the relative change of volume (or density) per unit pressure and is represented as

$$\beta_c = -\frac{dv/v}{dp} = \frac{d\rho/\rho}{dp}$$

Where,

dp = change in pressure

dv = change in volume

v = volume of the fluid

$d\rho$ = change in density

Note:

The negative sign indicates a decrease in volume v with increase in pressure.

1.4.2 Bulk modulus (*K*)

The bulk modulus of elasticity K is defined as the ratio of compressive stress to volumetric strain. Mathematically it is the reciprocal of compressibility (β_c).

$$K = \frac{1}{\beta_c} = \frac{dp}{\left(-\frac{dv}{v}\right)}$$

Unit: N/m^2 (Pa).

The bulk modulus of liquid and solid are the property of material. In other words, for a given liquid and solid, bulk modulus will be constant.

For gases, bulk modulus is not the property of material but depends on the process.

1.4.3 Compressibility, Bulk Modulus for an Ideal Gas

Process	Bulk Modulus of Elasticity (K)	Compressibility (β_c)
Isobaric process $p = C$	$K = 0$	$\beta_c = \infty$
Isochoric process $V = C$	$K = \infty$	$\beta_c = 0$
Isothermal process $T = C$	$K = p$	$\beta_c = \frac{1}{p}$
Isentropic process $pV^{\gamma} = C$	$K = \gamma p$	$\beta_c = \frac{1}{\gamma p}$

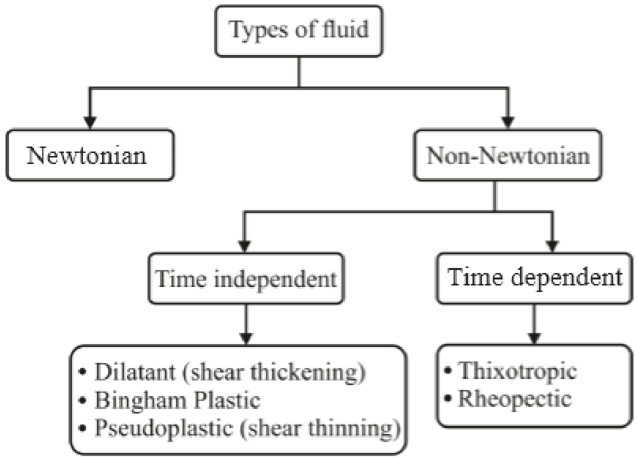
1.5 Types of Fluid

Ideal Fluid:

A fluid which has 0 viscosity, incompressible and zero surface tension is known as Ideal Fluid.

Real Fluid:

A real fluid is a fluid which has viscosity, finite compressibility and surface tension



1.5.1 Power Law Model:

$$\tau = A + B\left(\frac{du}{dy}\right)^n$$

Where

$A \geq 0$, is a constant representing Yield stress.

$B > 0$, is known as consistency Index.

n , is known as flow behavior Index.

$$\tau = A + \mu_{app}\left(\frac{du}{dy}\right), \left(\text{where } \mu_{app} = B\left(\frac{du}{dy}\right)^{n-1}\right)$$

μ_{app} = Apparent Viscosity

1.5.2 Dilatant Fluid (A = 0; n > 1):

- A fluid is said to be dilatant fluid if apparent viscosity increases with deformation rate.
- Dilatant fluid is also known as Shear – thickening fluid

Example:

Quick sand, Butter, Oobleck, Rice starch, Saturated sugar solution etc.

1.5.3 Pseudo Plastic Fluid (A = 0, n < 1):

- A fluid is said to be a pseudo plastic fluid, if apparent viscosity decreases with deformation Rate
- Pseudo plastic fluid is also known as shear – thinning fluid

Example:
Blood, Milk, Paper pulp, Water solution, Polymeric solution, Rubber, Molasses, Slurry etc.

1.5.4 Bingham Plastic Fluid (A > 0; n = 1):

It behaves as a rigid body at low stresses but flows as a viscous fluid at high stress.

- These fluids always have certain minimum shear stress before they yield.
- Shear strain rate increases then μ_{app} remains same.

Examples: Sewage sludge, Drilling mud, Tooth paste and Gel etc.

1.5.5 Rheopectic Fluid (A > 1; n > 1):

- For Rheopectic fluid apparent viscosity increases with time.

  Examples:
  Gypsum, Bentonite slurry, Whipping Cream

1.5.6 Thixotropic Fluid (A > 1; n < 1):

- For thixotropic fluid μ_{app} decreases with time.

  Examples: Lipstick, Printers ink, Hand lotion.

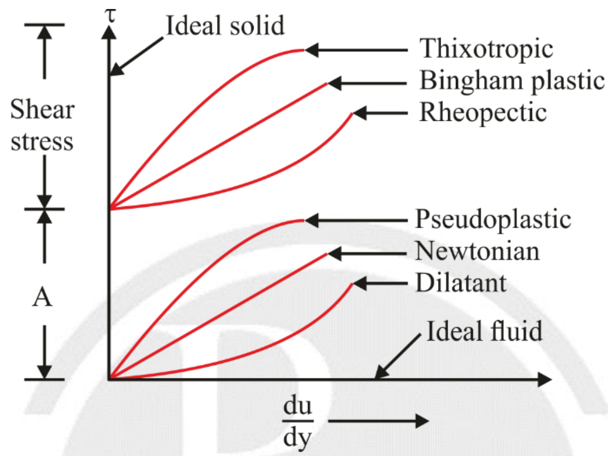


Fig. 1.7 Shear stress vs shear strain rate for difference fluids

1.6 Surface Tension (σ):

It is the tension of the surface film of a liquid caused by the attraction (cohesion) of the particles in the surface layer by the bulk of the liquid, which tends to minimize surface area.

$$\sigma = \frac{\text{Surface Tension Force}\left(F_{st}\right)}{\text{Length}\left(L\right)}$$

Units:

$\frac{N}{m}$ (or) $\frac{J}{m^2}$

Dimensional formula:

$[\sigma] = [MT^{-2}]$

Note:
- For ideal fluid, surface tension is zero due to absence of cohesion force.
- At critical point there is no distinct liquid vapour interface, hence surface tension is zero at critical point.
- On increasing temperature cohesion decreases, leading to decrease in surface tension.
- For water – air interface at 20°C its value is 0.0736 N/m and Air – mercury interface

  $\sigma = 0.480$ N/m.
- It is due to cohesion only.

1.6.1 Gauge Pressure inside Liquid Droplet:

$$\Delta p = \frac{4\sigma}{d}$$

Here, $\Delta p = p_{in} - p_{out}$ & $p_{out} = p_{atm} = 0\,\text{gauge}$

1.6.2 Gauge Pressure inside an Air Bubble

$$\Delta p = \frac{4\sigma}{d}$$

Here, $\Delta p = p_{in} - p_{out}$ & $p_{out} = p_{atm} = 0\,\text{gauge}$

1.6.3 Gauge Pressure inside A Soap Bubble:

$$\Delta p = \frac{8\sigma}{d}$$

Here, $\Delta p = p_{in} - p_{out}$ & $p_{out} = p_{atm} = 0\,\text{gauge}$

1.6.4 Gauge Pressure inside a Liquid Jet:

$$\Delta p = \frac{2\sigma}{d}$$

Here, $\Delta p = p_{in} - p_{out}$

1.6.5 Work Done in Stretching an Interface or Increase in Surface Energy:

$$W_d = \sigma \Delta A$$
$$W_d = \Delta E_s$$

Increase in surface energy $\Delta E_s = W_d = \sigma \Delta A$

$$E_s = \sigma A$$

Where ΔA is increase in Surface area of exposed Surface / interface area.

1.6.6 Work Done in Converting a large Spherical Droplet into 'n' Number of Small Spherical Droplet

$$W_d = \Delta E_s = \sigma\, 4\pi R^2 \left(n^{\frac{1}{3}} - 1\right)$$

Where R = Radius of large spherical droplet.

$r = R/n^{1/3}$

r = Radius of small spherical droplet.

1.7 Angle of contact or contact angle

Surface tension occurs during a gas liquid interface, but if that interface comes in contact with a solid surface such as the walls of a container the interface usually curves up or down near that surface. Such a concave or convex surface shape is known as meniscus.

The contact angle θ is determined as shown in the figure 1.8 and 1.9.

Case I When the meniscus is concave

- Adhesive force > cohesive force
- Angle of contact 'θ' is less than 90^0.

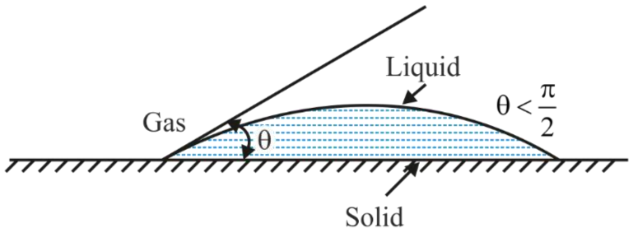


Fig.1.8 Wetting liquid (water) and solid surface interface

Case II When the meniscus is convex

- Adhesive force < cohesive force
- Angle of contact 'θ' is greater than 90°.

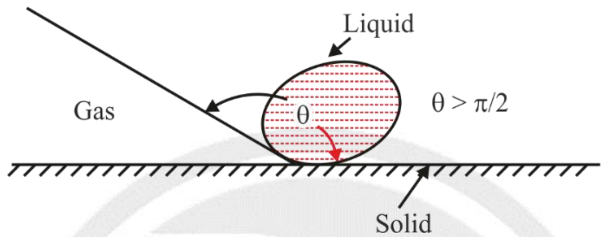


Fig.1.9 Non – wetting liquid (Mercury) and solid surface interface

Note:
- **(i) Cohesive Force:** Intermolecular attraction force between same molecules.
- **(ii) Adhesive Force:** Intermolecular attraction force between two different molecules.

1.8 Capillarity or Capillary Action

Capillarity or Capillary action is the ability of a narrow tube to rise or fall of a liquid against the force of gravity. Adhesion & cohesion both are responsible for capillary action. The rise of liquid surface is known as capillary rise while the fall of the liquid surface is known as capillary depression.

1.8.1 Capillary Rise

Adhesion > cohesion (Wetting liquid)

Vertically upward component of surface tension force = Weight of liquid rise

Capillary rise stops when vertically downward weight of the liquid rise is exactly balanced by the vertically upward component of surface tension force.

$$h = \frac{4\sigma \cos\theta}{\rho g D}$$

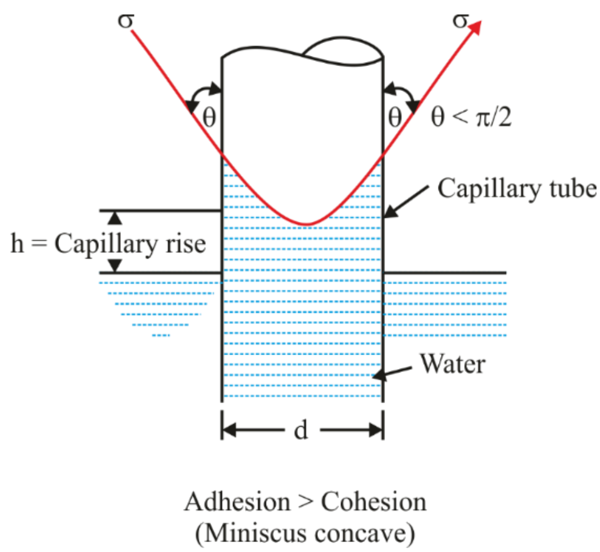


Fig.1.10 Capillary rise in a glass tube

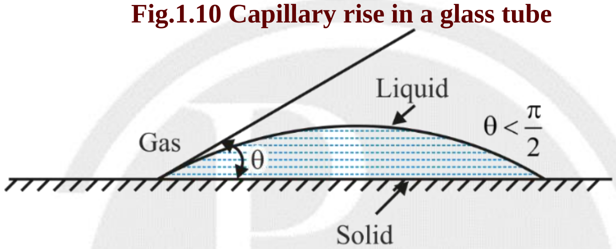


Fig.1.11 Wetting liquid (water) and solid surface interface

The most common interfaces and value of σ for clean interface at $20°C$ are

- $\sigma = 0.073\ \mathrm{N/m}$ for air-water interface
- $\sigma = 0.480\ \mathrm{N/m}$ for air-mercury interface
- $\theta =$ Angle of contact of the water surface
- For water-clean glass surface, $\theta = 0°$ and for mercury-clean glass, $\theta = 130°$

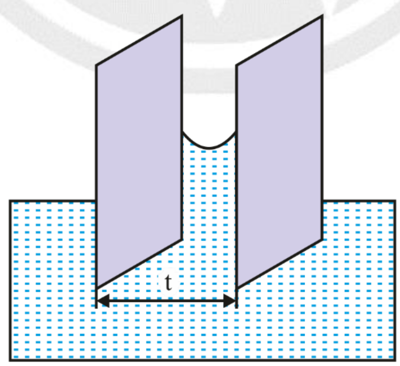


Fig.1.12 Capillary rise between two parallel plates

$$h = \frac{2\sigma \cos\theta}{\rho g t}$$

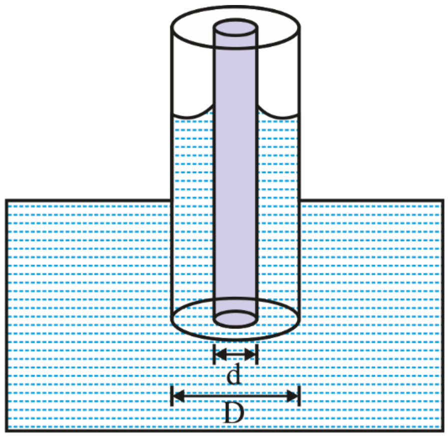


Fig.1.13 Capillary rise in the annulus of two concentric tubes

$$h = \frac{4\sigma\cos\theta}{\rho g(D-d)}$$

1.8.2 Capillary Fall

For Non-wetting Liquid, Cohesion > Adhesion and $\theta > 90°$

Capillary fall stops when vertically upward hydro – static force (pressure force) due to difference of liquid level is exactly balanced by vertically downward component of surface tension force.

$$h = -\frac{4\sigma\cos\theta}{\rho g d}$$

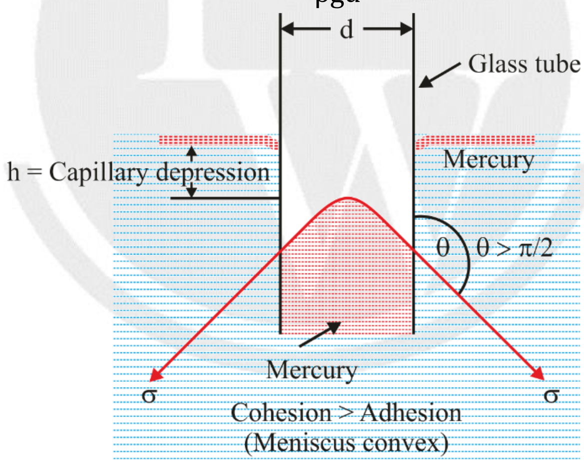


Fig.1.14 Capillary fall in the glass tube

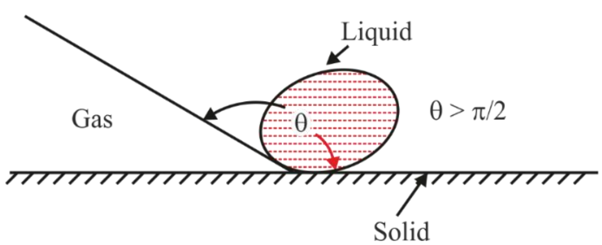


Fig.1.15 Non – wetting liquid (Mercury) and solid surface interface

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2 PRESSURE AND ITS MEASUREMENT

2.1 Introduction

Pressure is defined as the external force per unit area.

If dF is the force acting on the area dA in the normal direction then,

$$p = \frac{dF}{dA}$$

If the force (F) is uniformly distributed over the area (A), then,

$$p = \frac{F}{A} = \frac{Force}{Area}$$

Note:

(a) If there is no mass of fluid the pressure will be zero and it is known as Absolute Zero/Absolute vacuum.

(b) Characteristics of Pressure Force:

(i) External

(ii) Normal

(iii) Compressive

(c) Pressure is scalar quantity.

Units & Dimension of Pressure

Unit

$\frac{N}{m^2}$ or Pa

$1kPa = 10^3 Pa$

$1bar = 10^5\ Pa = 10^2\ kPa$

$1MPa = 10^6\ Pa = 10^3\ kPa = 10\ bar$

$[P] = [ML^{-1}T^{-2}]$

2.1.3 Absolute, Gauge, Atmospheric and Vacuum Pressures

1 Atmospheric Pressure (p_{atm})

Pressure exerted by atmosphere on a surface.

Atmospheric pressure at mean sea level:

$P_{atm,s}$ = 760 mm of Hg = 76 cm of Hg =760 Torr = 101.325 kPa = 1.01325 bar = 10.336 m of H_2O.

2 Gauge pressure (p_{gauge})

When pressure in a fluid is measured above the atmosphere pressure (p_{atm}) then it is called gauge pressure.

3 Vacuum pressure (p_v)

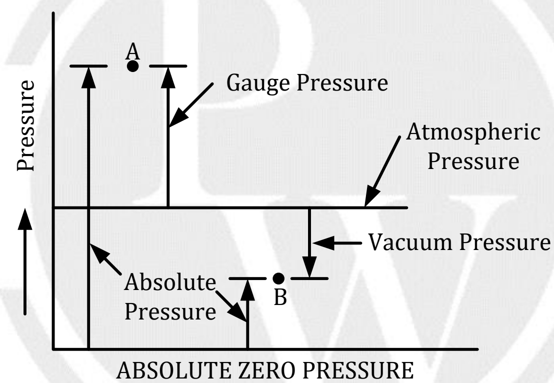
If the pressure at any point is below atmospheric then gauge pressure is negative which is called vacuum pressure.

$$p_v = -p_g$$

4 Absolute Pressure (p_{ab})

When pressure in a fluid is measured above the absolute zero or complete vacuum then it is called absolute pressure. Absolute pressure is always positive.

$$p_{ab} = p_{atm} + p_g$$



RELATIONSHIP BETWEEN PRESSURE

Fig.2.1 Absolute pressure, Gauge Pressure and Vacuum Pressure

2.2 Pascal's Law

Pressure at a point in a static fluid is equal in all directions.

- Pascal's law is valid for any static fluid.
- In the absence of shear forces, Pascal's law is applicable for a fluid in motion also.

2.3 Hydrostatic Law

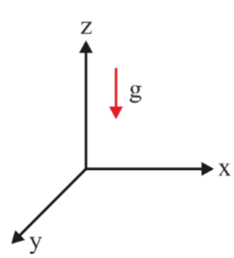


Fig.2.2

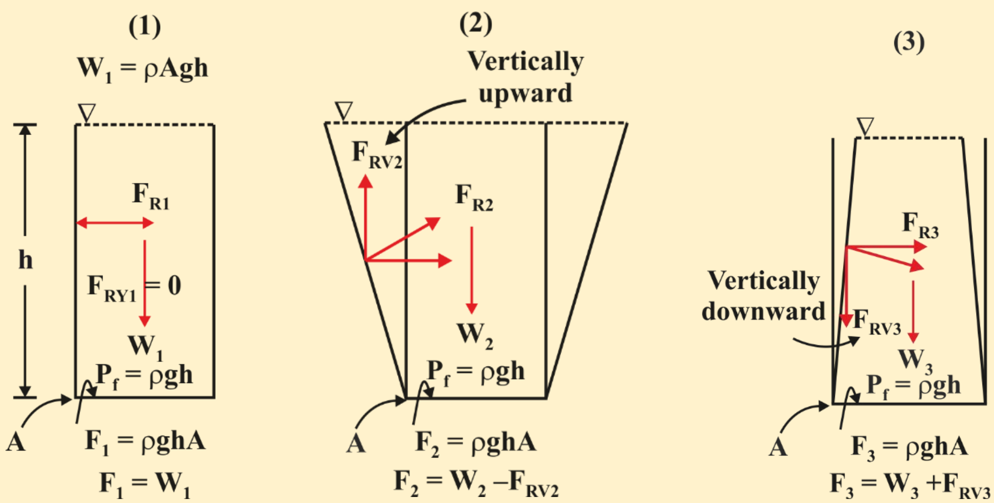
$$P = P(z) \text{ only}$$

$$\frac{dP}{dz} = -\rho g$$

- For static fluid, fluid pressure varies only in vertical direction ($i.e. in\ z$ direction) and magnitude of pressure gradient in vertical direction and is equal to the specific weight of the fluid. $\left(\because \frac{dp}{dz} = -\rho g\right)$
- Negative sign represents that pressure increases in vertically downward direction for a static fluid.

Note:

Fluid pressure is independent of the shape of the container it depends only on the depth of static incompressible fluid. This known as Hydrostatic Paradox.



Hear F = Net force

Fig.2.3 Comparison of pressure at the bottom and weight of the liquid in three containers.

2.4 Conversion of Liquid Column

$$\rho_1 h_1 = \rho_2 h_2$$

$$h_2 = \frac{\rho_1}{\rho_2} h_1$$

- Higher the density of liquid lesser is the depth of liquid required for a given fluid pressure.

Mercury has high density, hence requires lesser depth for a given fluid pressure. This is one of the reasons why mercury is used in manometer.

2.5 Barometer

Barometer is used to measure the local atmospheric pressure.

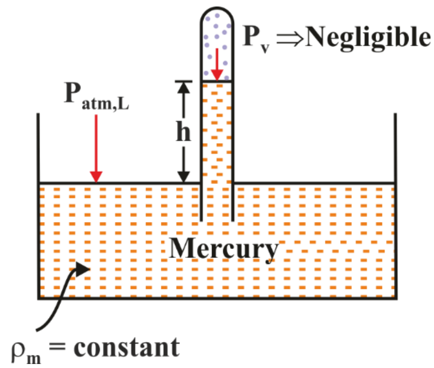


Fig.2.4

Measuring ⇒ Local Atmospheric Pressure

$P_{atm,L} = P_v + \rho_m gh$

Mercury ⇒ Low Vapour Pressure

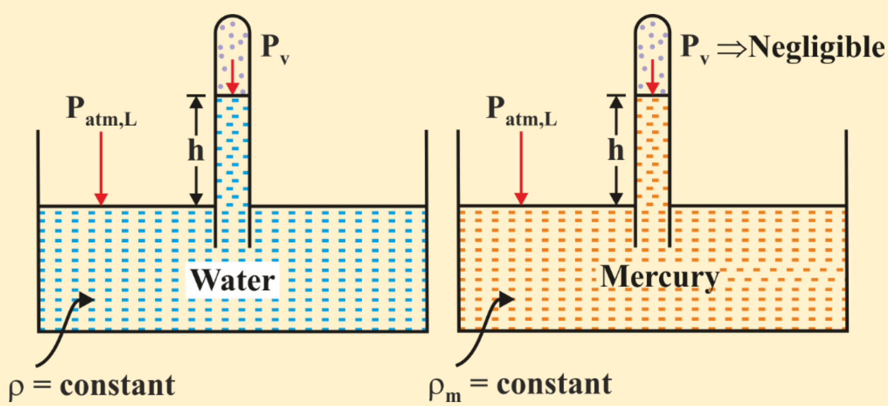
$P_{atm,L} \cong \rho_m gh$

Atmospheric pressure at mean sea level:

$P_{atm,s}$ = 760 mm of Hg = 76 cm of Hg =760 Torr = 101.325 kPa = 1.01325 bar = 10.336 m of H_2O

Note:

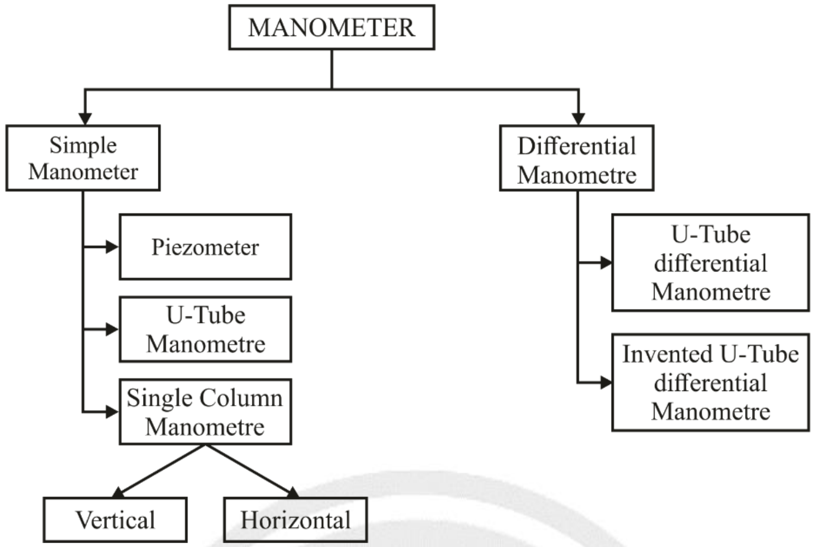
Barometer is used to measure local atmospheric pressure, Barometer was given by Torricelli (1608-1647). Local atmospheric pressure is measured by barometer hence local atmospheric pressure is also known as Barometric Pressure.



$P_{atm,L} = P_v + \rho gh$ $\qquad$ $P_{atm,L} \cong \rho_m gh$

Fig.2.5 Atmospheric pressure measurement by Barometer

2.6 Manometer



2.6.1 Piezometer

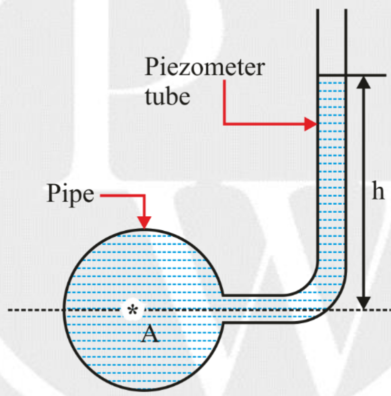


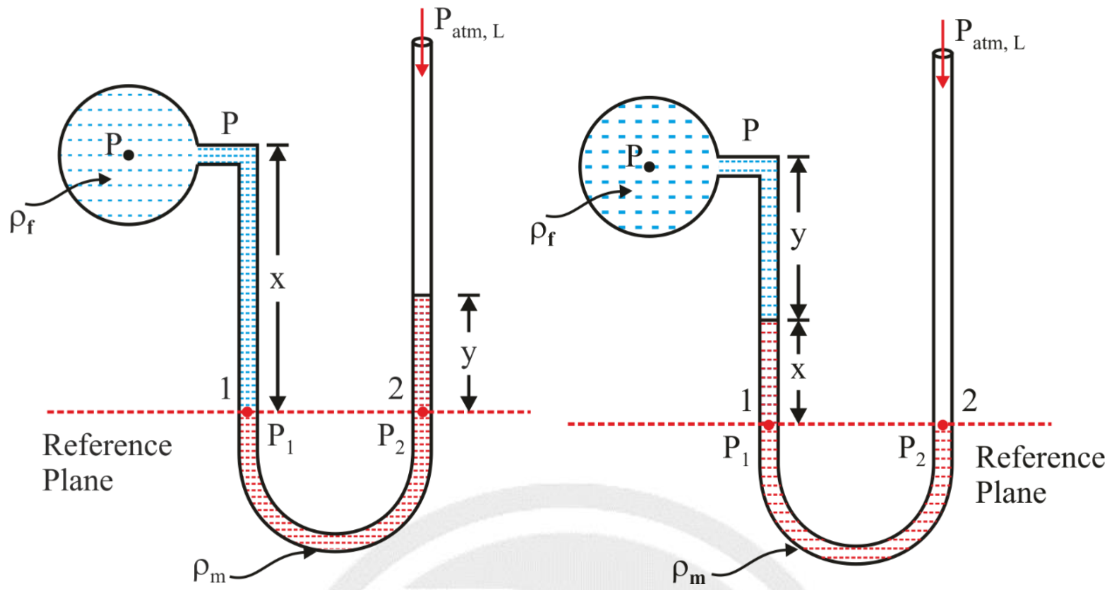
Fig.2.6 Piezometer

$$\left(p_g\right)_A = \rho_f gh$$

Limitations:

- Can't be used for gases
- Can't be used to measure vaccum pressure.
- Can't be used to measure high pressure.

2.6.2 U-Tube Manometer



$P_g = \rho_m gy - \rho_f gx$

$P_g = -(\rho_f gy + \rho_m gx) \Rightarrow P_v = \rho_f gy + \rho_m gx$

Fig.2.7 Pressure measurement by simple U – Tube Manometer

2.6.3 Vertical Single Column Manometer

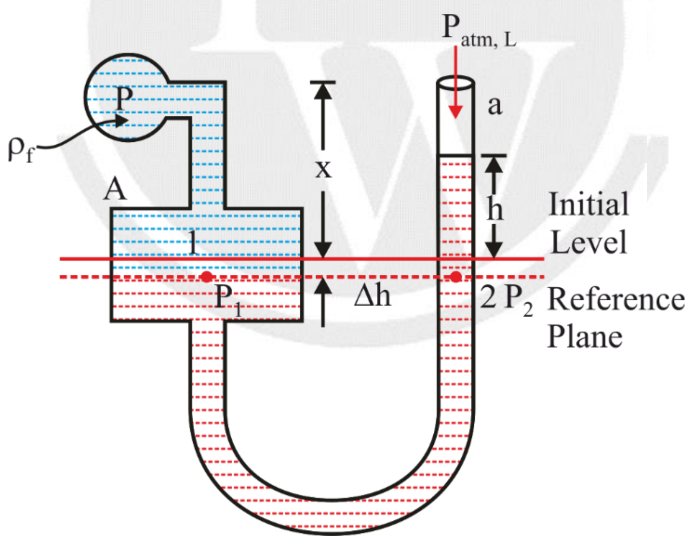


Fig.2.8 Pressure measurement by vertical single column manometer

Assumption: $P > P_{atm,L}$

$V_{fL} = V_{rR}$

$A \times \Delta h = a \times h$

$$\Delta h = \frac{a}{A} \times h$$

$\because$ $a \lll< A$

$\therefore \quad \Delta h \lll\lll< h$

- $P_g = \rho_m g(h + \Delta h) - \rho_f g(x + \Delta h) \rightarrow$ Exact value of gauge presure
- $P_g' = (\rho_m gh - \rho_f gx) \rightarrow$ Approximate value of gauge pressure
- $\%\text{ Error} = \frac{(\rho_m - \rho_f)g\Delta h}{\rho_m g(h+\Delta h) - \rho_f g(x+\Delta h)} \times 100 \rightarrow$ Exact value of % error
- $\%\text{ Error} = \frac{\Delta h}{h} \times 100 \rightarrow$ Approximate value of % error
- $\%\text{ Error} = \frac{a}{A} \times 100 \rightarrow$ Approximate value of % error

2.6.4 Inclined Single Column Manometer

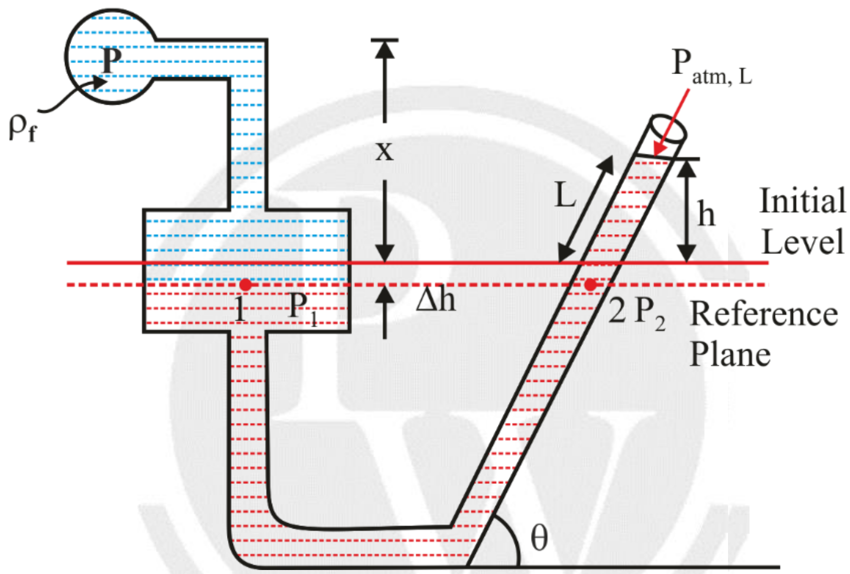


Fig.2.9 Pressure measurement by inclined single column manometer

$P_g = (\rho_m gL \sin\theta - \rho_f gx) \rightarrow$ Approximately

- Inclined single column manometer is used to measure low pressure.
- Using inclined limb, reading comes in terms of 'L' which is more than 'h' hence error decreases and sensitivity increases by $\frac{1}{\sin\theta}$.
- Sensitivity can also be increased by using lighter manometric fluid.

2.6.5 U-Tube Differential Manometer

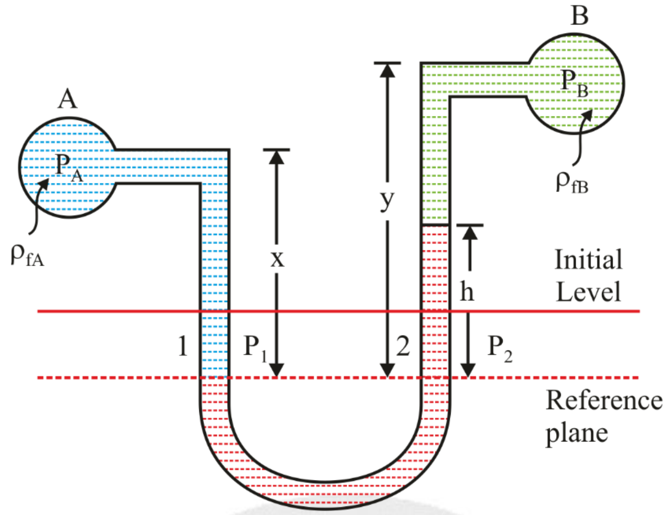


Fig.2.10 Pressure measurement by U – tube differential manometer

- Assumption: $P_A > P_B$
- $P_A - P_B = \rho_m gh + \rho_{fB} g(y - h) - \rho_{fA} gx$

> **Note:**
> Both pipes are at same level & Carrying Same Fluid.
> $P_A - P_B = (\rho_m - \rho_f)gh$

2.6.6 Inverted U-Tube Differential Manometer

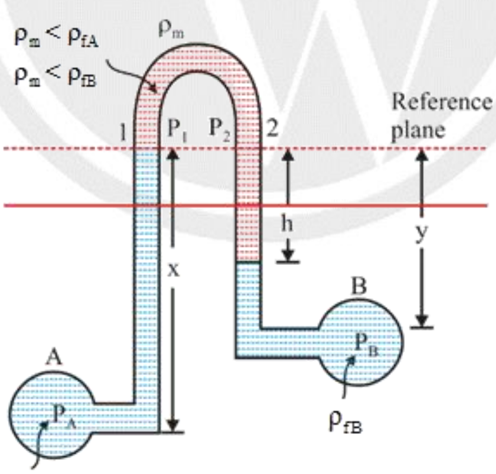


Fig.2.11 Pressure measurement by inverted U-tube differential manometer

- Assumption: $P_A > P_B$
- $\rho_m < \rho_{fA}$ & $\rho_m < \rho_{fB}$
- $P_1 = P_2$
- $P_A - P_B = \rho_{fA} gx - \rho_m gh - \rho_{fB} g(y - h)$

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3 HYDROSTATIC FORCE

3.1 Introduction

Hydrostatic Pressure force

It is defined as the force exerted by a static fluid on a surface either plane or curved when the fluid comes in contact with the surfaces. This force always acts normal to the surface.

3.2 Hydrostatic pressure on plane surface

3.2.1 Inclined Plane Surface

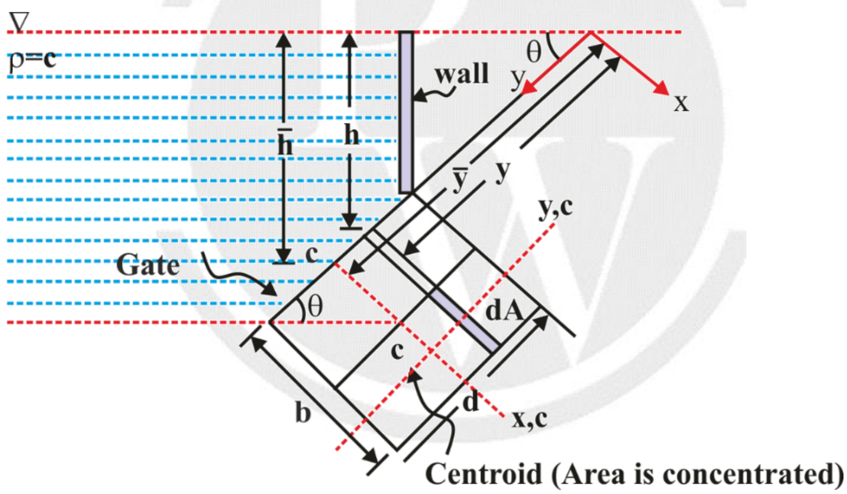


Fig:3.1 Hydrostatic force on a inclined plane surface

$$F_p = \rho g \bar{h} A$$

Here,

Here,

A = Area of surface which touching fluid

$\bar{h}$ = Vertical distance of C.O.G. of body from free surface.

θ = Angle at which the surface is inclined with horizontal

ρ = Density of liquid

F_p = Total pressure force on the surface

Note:

- Same expression can be used for vertical and horizontal plane surfaces also.
- Magnitude of $\bar{h}$ will depend on magnitude of angle of inclination (θ), hence magnitude of hydrostatic force will vary with angle of inclination.

Vertical Plane Surface (θ = 90°)

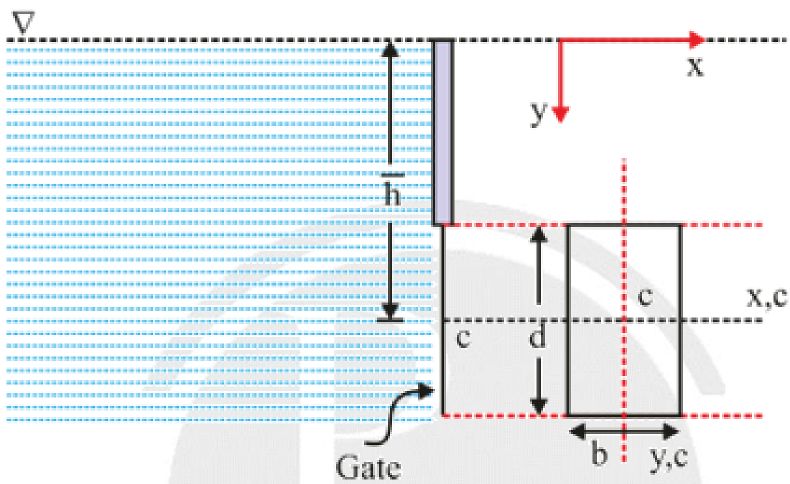


Fig:3.2 Hydrostatic force for a vertical plane surface

$$F_P = \rho g \bar{h} A$$

Horizontal Plane Surface (θ = 0°)

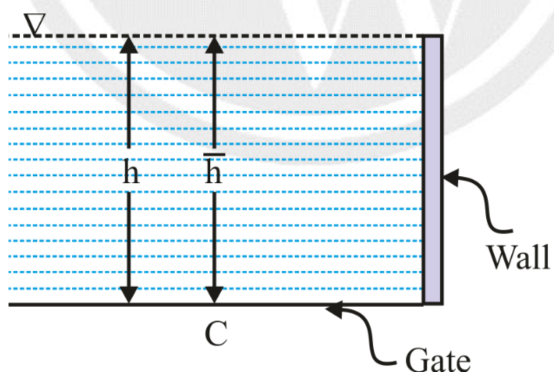


Fig:3.3 Hydrostatic force for a horizontal plane surface

$$F_P = \rho g \bar{h} A = \rho g h A$$

For horizontal plane surface, depth of centroid from the free surface $\bar{h}$ is equal to depth of horizontal plane surface from the free surface (h).

3.2.2 Centre of Pressure (h^*)

Centre of pressure is the point on the surface at which hydrostatic force (total pressure) is assumed to be acting.

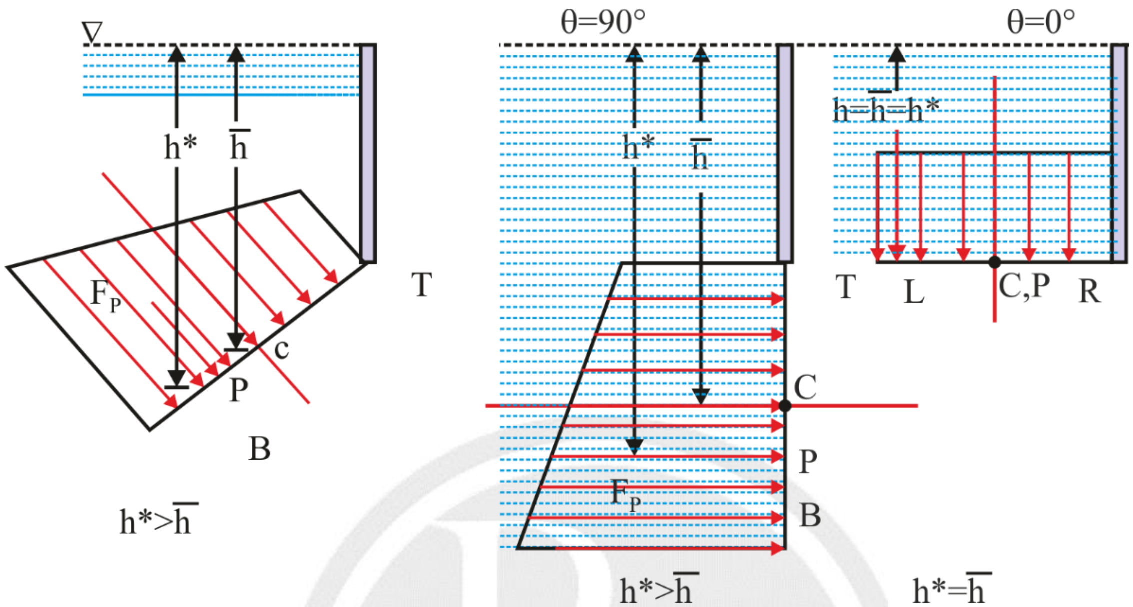


Fig:3.4 Centre of pressure under various conditions

For inclined plane surface

$$h^* = \bar{h} + \frac{I_{xx.c}\sin^2\theta}{A\bar{h}}$$

Here,

$I_{xx.c}$= Area moment of inertia about centroidal axis which is parallel to free surface.

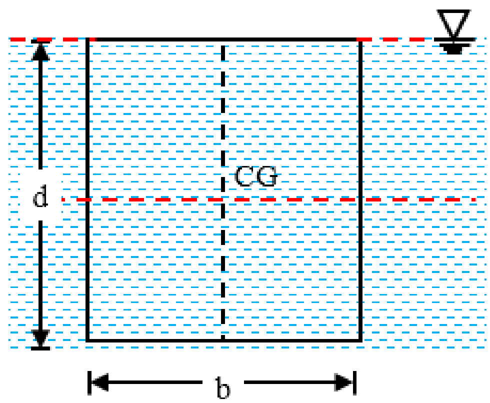
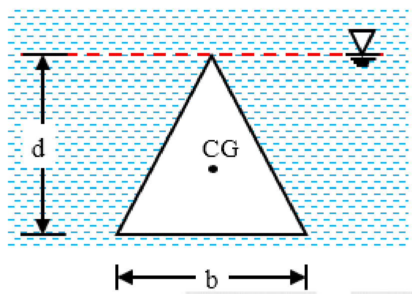
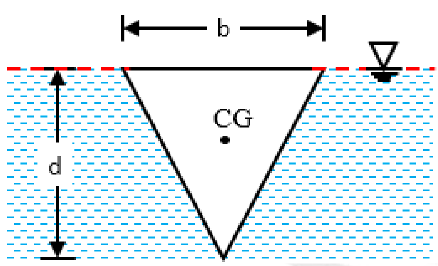
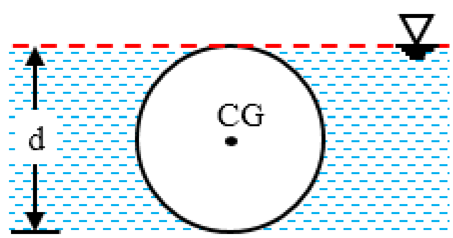
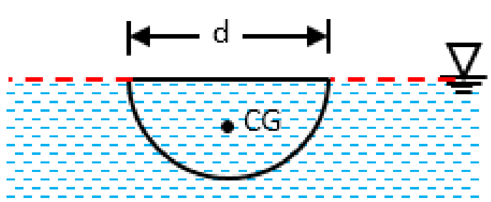
- For Vertical Plane Surface ($\theta = 90°$)

$$h^* = \bar{h} + \frac{I_{xx.c}}{A\bar{h}}$$

- For Horizontal Plane Surface ($\theta = 0°$)

$$h^* = \bar{h}$$

3.2.3 $\bar{h}$, h* AND M.O.I for Different Shapes

Surface	Shape	C.G. ($\bar{h}$)	M.O.I ($I_{xx.c}$)	C.P (h^*)
Rectangle		$\frac{d}{2}$	$\frac{bd^3}{12}$	$\frac{2d}{3}$
Triangle		$\frac{2d}{3}$	$\frac{bd^3}{36}$	$\frac{3d}{4}$
Inverted Triangle		$\frac{d}{3}$	$\frac{bd^3}{36}$	$\frac{d}{2}$
Circle		$\frac{d}{2}$	$\frac{\pi d^4}{64}$	$\frac{5d}{8}$
Semi-Circle		$\frac{2d}{3\pi}$	$\frac{\pi}{128}d^4 - \frac{\pi}{8}d^2\left(\frac{2d}{3\pi}\right)^2$	$\frac{3\pi d}{32}$

3.3 Force for Curved Surface

3.3.1 Horizontal Component (F_H)

- Horizontal component of hydrostatic force acting on a curved surface is equal to hydrostatic force acting on vertical projection of area.
- It will act at the centre of pressure of the vertical projection of curved area.

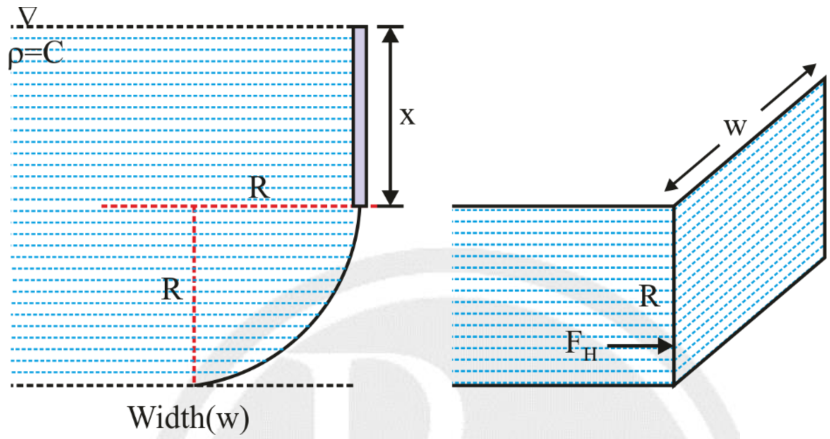


Fig:3.5 Horizontal component of hydrostatic force on curved surface

$$F_H = F_{p.v}$$

$$F_H = \rho g \bar{h}_v A_v$$

Here,

A_v = Projected area in vertical plane = $R \times w$ (for figure 3.5)

$\bar{h}_v$ = Vertical distance of C.O.G. of projected surface on vertical plane from free surface = $x + \frac{R}{2}$ (for figure 3.5)

Hence for figure 3.5,

$$F_H = \rho g\left(x + \frac{R}{2}\right)(Rw)$$

3.3.2 Vertical Component (F_V)

- Vertical component of hydrostatic force acting on a curved surface is equal to weight (W) of the fluid (Real or Imaginary) which can be contained above curved surface when the horizontal projection is taken up to free surface.
- It will be acting at the centre of gravity of the above-mentioned weight.

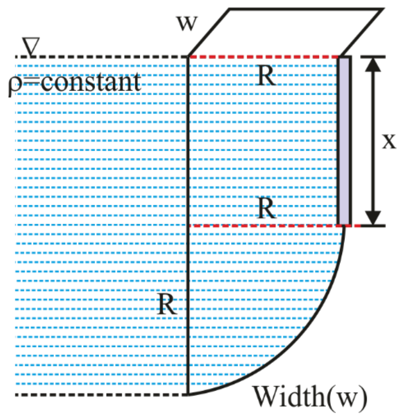


Fig: 3.6 Vertical component of hydrostatic force on curved surface

$$F_V = W$$

$$F_V = \rho V g$$

$$F_V = \rho\left(\frac{\pi}{4}R^2 w + Rxw\right)g$$

$$F_V = \rho\left(\frac{\pi}{4}R^2 + Rx\right)wg$$

3.3.3 Resultant Force (F_R)

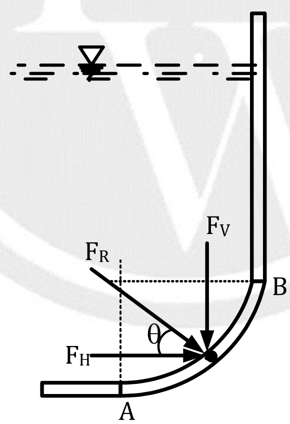


Fig:3.6

$$F_R = \sqrt{F_H^2 + F_V^2}$$

$$\tan\theta = \frac{F_V}{F_H}$$

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4 BUOYANCY AND FLOATATION

4.1 Introduction

- In the presence of gravity all liquids and gases exert an upward force on any object i.e. immersed on it, is known as force.

4.1.1 Buoyancy Force (F_B)

When a body is either partially or completely submerged in a fluid, there is a vertically upward force imposed by the fluid on the body. This force is known as Buoyant Force (F_B)

Buoyant Force ⇒ vertically upward direction

Archimedes Principle

Archimedes Principle helps to calculate the force of buoyancy. It states that the magnitude of force of buoyancy is equal to the weight of the fluid displaced by the body.

$F_b = W_{fd}$

$F_b = \rho_f V_{fd} g$

$F_b = \rho_f V_s g$

Here

V_{fd} = Volume of fluid displaced by the body

V_s = Volume of body immersed in fluid

W_{fd} = Weight of fluid displaced by the body

4.1.2 Centre of Buoyancy (B)

Centre of buoyancy is the point at which buoyant force is supposed to be acting. and it is the centroid of the volume of fluid displaced by the body

4.2 Condition of Floatation

A body will float in a liquid when weight of the body (W_b) is equal to the force of buoyancy & the line of action of force of buoyancy (F_b) & force of gravity must lie in the same vertical line.

Mathematically,

$F_b = W_b$

Note:

Solid body will float on the fluid, if density of the body is less than or equal to the density of fluid

4.3 Tension in Rope for Constrained Body ($\rho_b > \rho_f$)

At Equilibrium

$T + F_b = W_b$

$T = W_b - F_b$

$T = \left(\rho_b V_b - \rho_f V_{f,d}\right) g$

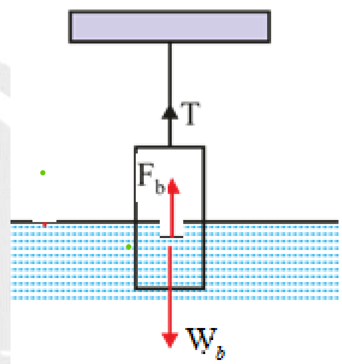

Fig: 4.1 Weight loss due to Buoyancy

4.4 Body Completely Submerged in two Liquids

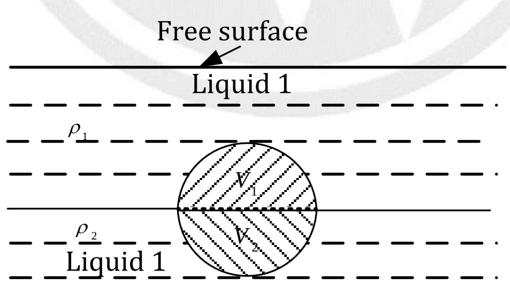

Fig: 4.2 Buoyant force on a body submerged in two liquids

Force of buoyancy,

$$F_B = \rho_1 g V_1 + \rho_2 g V_2$$

Where,

ρ_1 = density of fluid 1

ρ_2 = density of fluid 2

V_1 = Volume of part of body immersed in liquid 1

V_2 = Volume of part of body immersed in liquid 2.

4.5 STABILITY OF SUBMERGED & FLOATING BODIES

4.5.1 Type of equilibrium

(A) Stable Equilibrium

It occurs when a body is tilted slightly by some external force, and then it returns back to its original position due to the weight and the up thrust.

(B) Unstable Equilibrium

It occurs when a body is tilted slightly by some external force & body does not return to its original position from the slightly displaced angular position.

(C) Neutral Equilibrium

It occurs when a body, when given a small angular displacement, occupies a new position and remains at rest.

4.5.2 Rotational Stability of Completely Submerged Body

In case of completely submerged body the position of center of gravity (G) & center of buoyancy is fixed.

Position of G & B determines the stability of submerged bodies:

Case 1: G lies below B ⇒ Stable Equilibrium

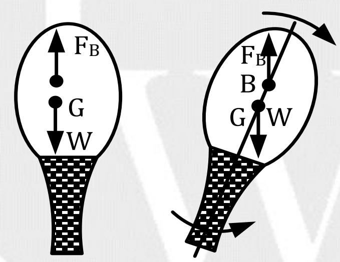

Fig: 4.3 Stable equilibrium

Case 2: G lies above B ⇒ Unstable Equilibrium

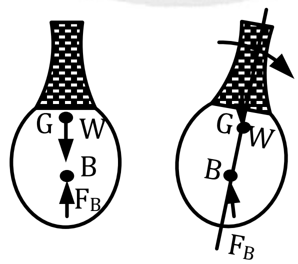

Fig: 4.4 Unstable equilibrium

Case 3: G coincides with B ⇒ Neutral Equilibrium

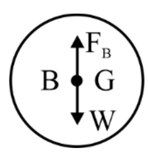

Fig: 4.5 Neutral equilibrium

4.5.3 Stability of Completely Floating Body

Meta-centre

If a body which is floating in liquid is given small angular displacement, it starts oscillating about some point M. This point is called meta-center. It can also be defined as the point at which the line of action of force of buoyancy will meet the normal axis of the body when body is given a small angular displacement.

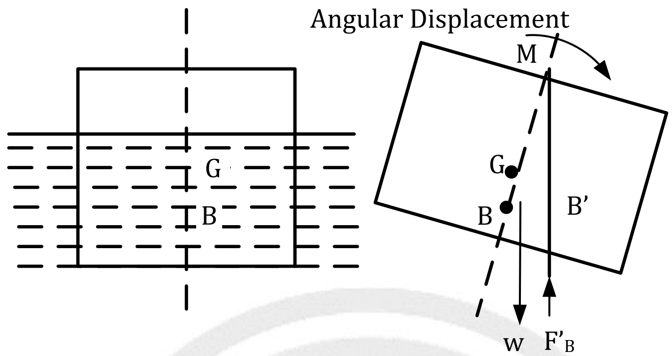

Fig: 4.6 Meta-center

Metacentric height (GM)

Distance between Metacenter of floating body and the center of gravity of the body is called Metacentric Height.

$$GM = BM - BG$$

Expression for calculating metacentric height:

Metacentric height,

$$GM = BM - BG$$

$$GM = \frac{I_{min}}{V_d} - BG$$

Where,

GM = Metacentric Height

BM = Distance between center of buoyancy under equilibrium (B) & Meta-center (M) = Metacentric radius

BG = Distance between center of buoyancy under equilibrium (B) & center of gravity of body (G)

I = Moment of inertia of top view of floating body about axis of rotation.

= Minimum moment of inertia of top view at the free surface

V_d = Volume of fluid displaced

Rotational stability of floating body

Case I: If the point M is above G (i.e. Metacentric Height is +ve), the body is in stable equilibrium

Case II: If the point M is below G (i.e. Metacentric Height is -ve), the body is in unstable equilibrium

Case III: If the point M coincides with G (i.e. Metacentric Height is 0), the body is in neutral equilibrium.

4.6 Time Period of Oscillation

$$T = 2\pi\sqrt{\frac{k^2}{gGM}}$$

k = radius of gyration of floating body

$$GM_2 > GM_1$$

$$T_2 < T_1$$

Increasing the metacentric height gives greater stability to a floating body and reduces time period of rolling of the body.

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5 FLUID KINEMATICS

5.1 Introduction

Fluid Kinematics deals with the motion of Fluid without considering cause of motion.

5.1.1 Lagrangian approach

Lagrangian approach is for single Fluid particle. (Closed system analysis)

In Lagrangian approach fluid motion is defined by analysing the kinematic behaviour of each and every individual fluid particle consisting the flow.

$\vec{S} = \vec{S}\left(\vec{S_0}, t\right)$

Here, $\vec{S} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{S_0} = x_0\hat{i} + y_0\hat{j} + z_0\hat{k}$, $t = \text{time}$

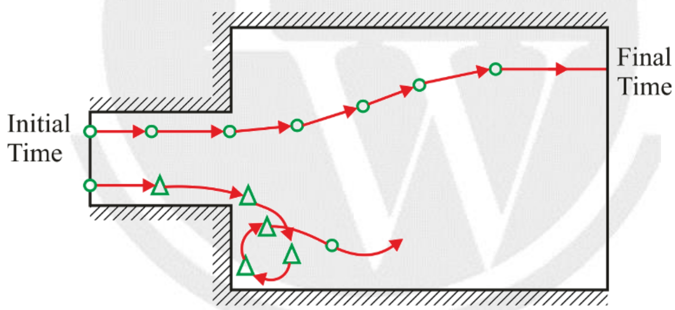

Fig.5.1 (Lagrangian Approach)

5.1.2 Eulerian approach

Eulerian approach for particular section (or) point. (Open system analysis)

In Eulerian approach fluid motion is defined by analysing the kinematic behaviour of various fluid particles passing through the various fixed points in a control volume.

$\vec{V} = \vec{V}\left(\vec{S}, t\right)$

Here $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$, $\vec{S} = x\hat{i} + y\hat{j} + z\hat{k}$ and $t = \text{time}$

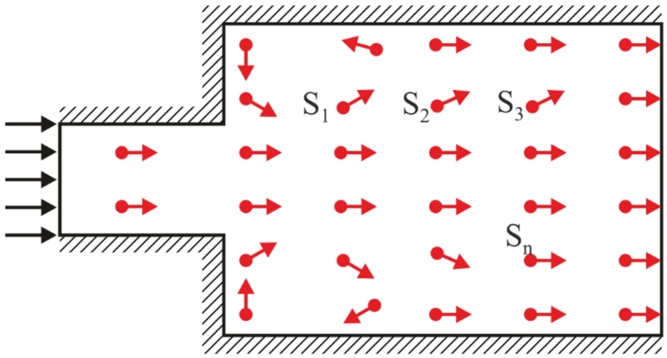

Fig.5.2 Eulerian Approach
(Study of particular section with time)

5.2 Velocity Vector

Velocity Vector in Cartesian Coordinates

Let *u*, *v*, *w* are component of velocity in *x*, *y*, *z* direction.

$$\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$$

Here
$$u = u(x, y, z, t)$$
$$v = v(x, y, z, t)$$
$$w = w(x, y, z, t)$$
$$|\vec{V}| = \sqrt{|u|^2 + |v|^2 + |w|^2}$$

Velocity Vector in Cylindrical Coordinates

$$\vec{V} = v_r\hat{r} + v_\theta\hat{\theta} + v_z\hat{z}$$

$$v_r = \frac{dr}{dt} \text{ and } v_\theta = \frac{rd\theta}{dt}, \; v_z = \frac{dz}{dt}$$

5.3 Acceleration Vector ($\vec{a}$)

$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

$$|a| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

5.3.1 Convective/Advective Acceleration (a_c)

The part of acceleration which is due to change in velocity components with space (x, y, z) is known as Convective/Advective acceleration.

$$a_{c,x} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

5.3.2 Local/Temporal Acceleration (a_l)

The part of acceleration which is due to change in velocity components with time (t) is known as Local/Temporal Acceleration.

$$a_{l,x} = \frac{\partial u}{\partial t}$$

Note:

- Total Acceleration = Convective acceleration + local acceleration

5.3.3 Acceleration Vector in Cylindrical Coordinates

$$a_r = v_r\frac{\partial v_r}{\partial r} + v_\theta\frac{\partial v_r}{r\,\partial\theta} + v_z\frac{\partial v_r}{\partial z} + \frac{\partial v_r}{\partial t} - \frac{v_\theta^2}{r}$$

$$a_\theta = v_r\frac{\partial v_\theta}{\partial r} + v_\theta\frac{\partial v_\theta}{r\,\partial\theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{\partial v_\theta}{\partial t} + \frac{v_r v_\theta}{r}$$

$$a_z = v_r\frac{\partial v_z}{\partial r} + v_\theta\frac{\partial v_z}{r\,\partial\theta} + v_z\frac{\partial v_z}{\partial z} + \frac{\partial v_z}{\partial t}$$

5.4 Various Types of Fluid Flow

(i) Steady and Unsteady flow
(ii) Uniform and Non-Uniform flow
(iii) Compressible and Incompressible flow
(iv) Laminar and Turbulent flow
(v) Rotational and Irrotational flow
(vi) 1-D, 2-D & 3-D flow.

5.4.1 Steady and Unsteady Flow

A flow is said to be steady flow if fluid properties & flow velocity does not change with time at given location.

The flow which is not steady flow is called unsteady flow.

Steady Flow in Eulerian Approach

For a steady flow, acceleration in a given direction is equal to the convective acceleration in that particular direction, as local acceleration is zero.

$$\vec{V} = \vec{V}(\vec{S}, t) \qquad \text{(In general)}$$

For steady flow

$$\vec{V} \neq \vec{V}(t)$$

$$\therefore\ \vec{V} = \vec{V}(\vec{S}) \qquad \text{only}$$

5.4.2 Uniform and Non-Uniform Flow

A flow is said to be a uniform flow if fluid properties & flow velocity are does not change with space at any given instant of time.

The flow which is not uniform flow is called non-uniform flow.

Uniform Flow in Eulerian Approach

For a uniform flow, acceleration in a given direction is equal to the local acceleration in that particular direction as convective acceleration is zero.

$$\vec{V} \neq \vec{V}(\vec{S}) \text{ and } \vec{V} = \vec{V}(t) \text{ only}$$

Type of fluid flow	Convective Acceleration	Temporal Acceleration	Total Acceleration
Unsteady & Non-uniform	Exists	Exists	Exists
Unsteady & Uniform	Zero	Exists	Exists
Steady & Non-Uniform	Exists	Zero	Exists
Steady & Uniform	Zero	Zero	Zero

5.4.3 Compressible and Incompressible Flow

A flow is said to be incompressible flow if mathematically total derivative/material derivative of density is zero during the flow.

$$\frac{D\rho}{Dt} = 0$$

The flow which is not incompressible flow is compressible flow

Flow of compressible fluid can be incompressible flow, if Mach number must be less than equal to 0.3

5.4.4 Laminar & Turbulent Flow

1. Laminar flow

It is defined as the type of flow in which fluid particle move along well defined path or stream line.

2. Turbulent Flow

It is defined as the type of flow in which fluid particle move in Zig-zag way or random order.

5.4.5 Rotational and Irrotational Flow

A flow is said to be rotational flow if the fluid particles rotate about its own centre of mass during the motion otherwise flow is irrotational.

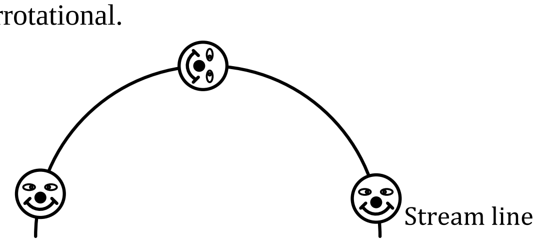

Fig.5.3 Rotaional Flow

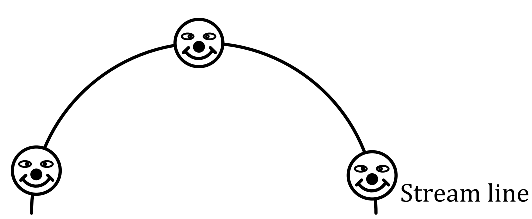

Fig.5.4 Irrotaional Flow

A flow will be Irrotational under the following two conditions:

(i) Ideal fluid flow (inviscid fluid flow)

(ii) Real fluid flow having negligible velocity gradient.

5.4.6 1-D, 2-D & 3-D Flow

A flow is classified as 1-D, 2-D or 3-D flow depending on the number of space coordinates required to specify the velocity field.

5.5 Various Flow Lines

- Streamline
- Path line
- Streak line
- Time line

5.5.1 Streamline

Streamline is an imaginary line or curve in the space such that tangent drawn to it at any point gives the direction of instantaneous velocity of fluid particle present at that point.

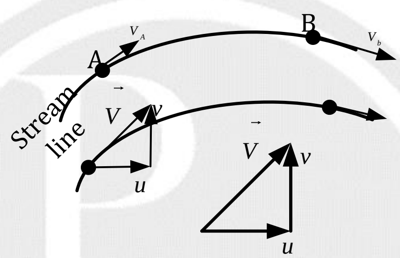

Fig.5.5 Stream lines

Note:

- Component of velocity in normal direction to the streamline is zero, hence there is no flow across the streamline.
- Streamline is drawn for an instant of time.
- Streamlines are based on Eulerian approach.
- Two streamlines cannot intersect each other.
- A given streamline cannot intersect itself.
- Streamlines are defined everywhere except at stagnation point.
- A bundle of neighbouring streamlines may be imagined to form a passage through which the fluid flows. This passage is known as stream tube.

Differential Equation of Streamline in Cartesian Coordinate

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

5.5.2 Path Line

- Path line is the path traced by single fluid particle at different instants of time.
- Path line is based on Lagrangian approach.
- A path line can intersect itself.

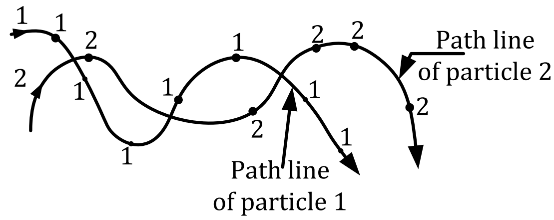

Fig.5.6 Path lines

Stream line	Path line
Direction of Instantaneous velocity	Position / Location
Number of fluid particles	Single fluid Particle
Particular instant of time	Over a period of time
Eulerian approach	Lagrangian approach
Can't intersect itself	Can intersect itself

5.5.3 Streakline

- Streakline is the locus of different fluid particles which passes through a given point.
- Streakline is drawn for a particular instant of time.

Note:
For a steady flow, streamline, path line and streak line are identical.

5.5.4 Timeline

- A timeline is a set of adjacent fluid particles in a flow field that were marked at a given instant of time.

5.6 Volume Flow Rate (Q)

Volume of fluid flowing through a cross section per unit time is known as volume flow rate.

$$Q = \int_{A_c} dA_c \times V_n$$

Units: m^3/s
Dimensional formula:

$$[Q]=[L^3T^{-1}]$$

5.7 Mass Flow Rate

$$\delta\dot{m} = \rho V_n dA$$

$$\dot{m} = \int_A \delta\dot{m}$$

If density is contant,

$$\dot{m} = \rho\dot{Q}$$

5.8 Concept of Average Velocity (V)

Average velocity is the hypothetical uniform velocity across the cross section, which gives same mass flow rate as that of actual mass flow rate through the cross section based on actual velocity.
Average velocity ⇒ Hypothetical uniform velocity

$$\dot{m}_{act} = \dot{m}_{avg}$$

$$\dot{m}_{av} = \rho A_c V$$

- According to the definition of Average Velocity $V = \frac{1}{A_c}\int_{A_C} u dA_c$ (Any cross – section)
- For pipe flow $V = \frac{2}{R^2}\int_0^R urdr$

5.9 Continuity Equation

- It is also known as conservation of mass

$$\dot{m}_i = \dot{m}_o + \dot{m}_{cv}$$

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) + \frac{\partial \rho}{\partial t} = 0$$

$$\nabla.\left(\rho\vec{V}\right) + \frac{\partial \rho}{\partial t} = 0 \qquad \text{(For any flow)}$$

5.9.1 Special Case 1: Steady Flow

- For steady flow $\frac{\partial \rho}{\partial t} = 0$

$$\therefore \quad \nabla.\left(\rho\vec{V}\right) = 0$$

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

5.9.2 Special Case 2: Incompressible Flow

$$\nabla.\vec{\nabla} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

- For an incompressible flow divergence of velocity vector is zero.

5.9.3 Special Case: For 1 D Steady flow

$$\rho AV = \text{Constant}$$

- For incompressible fluid flow

$$AV = \text{Constant}$$

5.10 Strain Rate

5.10.1 Linear Strain Rate ($\dot{\varepsilon}_{\forall}$)

It is defined as the rate of change of length per unit length.

$$\dot{\varepsilon}_{xx} = \frac{\partial u}{\partial x}, \quad \dot{\varepsilon}_{yy} = \frac{\partial v}{\partial y}, \quad \dot{\varepsilon}_{zz} = \frac{\partial w}{\partial z}$$

5.10.2 Volume Strain Rate ($\dot{\varepsilon}_{\forall}$)

The rate of change in volume per unit volume

$$\dot{\varepsilon}_{\forall} = \frac{1}{\forall}\frac{D\forall}{Dt} = \nabla \cdot \vec{V}$$

- It is also known as Dilation rate

$$\dot{\varepsilon}_{\forall} = \dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy} + \dot{\varepsilon}_{zz}$$

$$\dot{\varepsilon}_{\forall} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

- For Incompressible flow

$$\dot{\varepsilon}_{\forall} = 0$$

5.10.3 Shear Strain Rate

$$\dot{\varepsilon}_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, \quad \dot{\varepsilon}_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \quad \dot{\varepsilon}_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

5.11 Condition for Pure Rotation (Zero Angular Deformation)

For pure rotation, angular deformation is zero

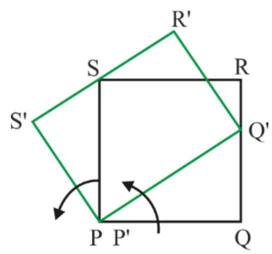

Fig.5.7

For 2 − D (x − y plane) Flow

$$\dot{\varepsilon}_{xy} = 0$$

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0$$

5.12 Angular Velocity Vector

$$\omega = \frac{1}{2}\left(Curl\left(\vec{V}\right)\right) = \frac{1}{2}\vec{\nabla}\times\vec{V}$$

$\Rightarrow$ Angular velocity vector is half of curl of velocity vector.

$$\vec{\omega} = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$\omega_{xy} = \omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\omega_{yz} = \omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\omega_{zx} = \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

Condition for Irrotational Flow in 2-D (x-y plane)

- For irrotational flow, $\nabla \times \vec{V} = \vec{0}$

$$\omega_z = 0$$

$$\therefore \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

- For irrotational flow, curl of velocity vector is zero vector

5.12.1 Condition for Zero Rotation

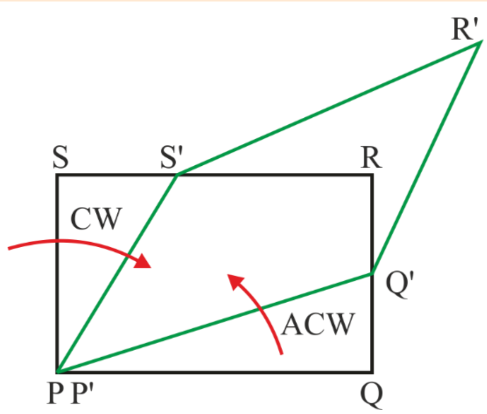

Fig.5.8

- For 2 – D(x – y plane) Flow

$$\omega_{xy} = 0$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

5.13 Vorticity Vector $(\vec{\Omega})$

Curl of velocity vector is known as vorticity vector.

$$\vec{\Omega} = 2\vec{\omega}$$

(Vorticity vector is twice of the angular velocity vector of the fluid particle)

$$\vec{\Omega} = \nabla \times \vec{V}$$

- For Irrotational Flow vorticity vector is null vector

$$\vec{\Omega} = 2\vec{\omega} = \vec{0}$$

$$\nabla \times \vec{V} = \vec{0}$$

Vorticity is a measure of rotation of a fluid particle.

5.14 Circulation (Γ)

Circulation is defined as the line integral of the tangential component of the velocity vector taken around a closed curve

Circulation represents the strength of a rotational flow.

$$\Gamma = \oint_C \vec{V}.\vec{dr}$$

$$\Gamma = \Omega A \text{ (Vorticity x inclosed area)}$$

'C'(anti clock wise) in the flow field.

5.15 Velocity Potential Function (ϕ)

Velocity potential function is defined as the scalar function of space and time such that the partial differentiation with respect to any direction gives the component of velocity in that direction for irrotational flow.

Mathematically,

$$u = \frac{\partial \phi}{\partial x}, v = \frac{\partial \phi}{\partial y}, w = \frac{\partial \phi}{\partial z}$$

Some important point:

(i) Existence of velocity potential function implies that flow is irrotational.

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial^2\phi}{\partial x \partial y} - \frac{\partial^2\phi}{\partial x \partial y}\right)$$

$$\omega_z = 0$$

Similarly,

$$\omega_x = 0 \text{ and } \omega_y = 0$$

Since $\omega_x = \omega_y = \omega_z = 0$, hence flow will be irrotational therefore we can conclude that existence of velocity potential function implies that flow is irrotational.

(ii) If velocity potential function satisfies Laplace equation, the incompressible flow is possible.

Differentiating equation 1 with respect to x, y and z direction respectively,

$$\frac{\partial u}{\partial x} = \frac{\partial^2\phi}{\partial x^2}, \frac{\partial v}{\partial x} = \frac{\partial^2\phi}{\partial y^2}, \frac{\partial w}{\partial x} = \frac{\partial^2\phi}{\partial z^2} \qquad \ldots\text{(ii)}$$

Laplace equation is given by

$$\nabla^2\phi = 0$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad \ldots\text{(iii)}$$

From equation (ii) and (iii),

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Above equation is continuity equation for incompressible fluid. Hence flow will be incompressible possible flow when potential function satisfies Laplace equation.

(iii) Velocity potential function (ϕ) is exact function or point function:

Let, Velocity potential function (ϕ) is two dimensional, i.e.

$$\phi = f(x, y)$$

Since ϕ is exact function, therefore,

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy$$

And $d\phi$ will be exact differential, therefore comparing $d\phi$ with exact differential, $d\phi = Mdx + Ndy$,

$$M = \frac{\partial\phi}{\partial x} \text{ and } N = \frac{\partial\phi}{\partial y}.$$

And if $M \& N$ is known then ϕ is determine by the expression given below:

$$\boxed{\phi = \left[\int_{y=\text{const.}} Mdx + \int (\text{terms of } N \text{ nor containg } x)\, dy\right]}$$

(iv). Equipotential lines

A line along which, velocity potential function (ϕ) remains constant, is called equipotential line.

Slope of equipotential line for 2-D flow $\left(\frac{dy}{dx}\right)$:

For 2D flow,

$$\phi = f(x, y)$$

Since ϕ is exact function, therefore,

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy$$

Since, for equipotential line, $\phi = \text{constant} \Rightarrow d\phi = 0$

Therefore, from above equation

$$0 = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}} \qquad \left[\because u = \frac{\partial \phi}{\partial x}, v = \frac{\partial \phi}{\partial y}\right]$$

$$\Rightarrow \quad \boxed{\frac{dy}{dx} = -\frac{u}{v}} \qquad \text{(Except stagnation point)}$$

5.15.1 Velocity Potential function in Cylindrical Coordinate system

For irrotational flow, the velocity potential function in cylindrical coordinates are.

$$\frac{\partial \phi}{\partial r} = V_r$$

$$\frac{\partial \phi}{r\,\partial \theta} = V_\theta$$

$$\frac{\partial \phi}{\partial z} = V_z$$

5.16 Stream Function Ψ

It is defined as the scalar function of space and time such that its differentiation with respect to any axis gives the component of velocity at right angle to this axis. It exists only for 2-D flow.

Mathematically,

$$u = \frac{\partial \psi}{\partial y} \;\&\; v = -\frac{\partial \psi}{\partial x}$$

Where u, v are component of velocity in x, y direction respectively

Some important point:

(i) Existence of stream function signifies that the flow is incompressible and possible.

From equation (i)

$$\frac{\partial u}{\partial x} = \frac{\partial^2 \psi}{\partial x \partial y}, v = \frac{-\partial^2 \psi}{\partial x \partial y}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

The above equation is continuity equation for 2-D incompressible flow. Hence, we can conclude that existence of stream function signifies that the flow is incompressible and possible.

(ii) If Stream function satisfies the Laplace equation, then flow will be irrotational.

Laplace equation for 2-D is given by

$$\nabla^2 \psi = 0$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

$$-\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0$$

$$\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

$$\omega_z = 0$$

Since $\omega_z = 0$ is zero (i.e. flow is irrotational) when stream function satisfying the laplace equation, hence we can conclude that flow will be irrotational if stream function satisfies the laplace equation otherwise flow will be rotational.

(iii) The flow rate per unit depth between two stream line (S_1 & S_2) is constant & given by,

$$Q = \psi_1 - \psi_2$$

Where,

$Q =$ flow rate per unit depth between to stream line (S_1 & S_2)

ψ_1 = equation of stream line S_1

ψ_2 = equation of stream line S_2

(iv) Stream function (ψ) is exact function or point function:

Let, $\psi = f(x, y)$

Since ψ is exact function, therefore,

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

And $d\psi$ will be exact differential, therefore comparing $d\psi$ with exact differential, $d\psi = Mdx + Ndy$,

$$M = \frac{\partial \psi}{\partial x} \text{ and } N = \frac{\partial \psi}{\partial y}.$$

And if M and N is known then ψ is determine by the expression given below:

$$\boxed{\psi = \left[\int_{y=\text{const.}} M dx + \int (\text{terms of } N \text{ nor containg } x) \right]}$$

(v) **Stream line**

Line along which stream function is constant is known as stream line.

Slope of stream line for 2-D flow $\left(\frac{dy}{dx}\right)$,

For 2D flow,

$$\psi = f(x, y)$$

Since ψ is exact function, therefore,

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

Since, for stream line, ,$\psi = \text{constant} \Rightarrow d\psi = 0$

Therefore, from above equation

$$0 = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial y}} \qquad \left[\because u = \frac{\partial \psi}{\partial y} \,\&\, v = -\frac{\partial \psi}{\partial x} \right]$$

$$\Rightarrow \quad \boxed{\frac{dy}{dx} = \frac{v}{u}} \qquad \text{(Except stagnation point)}$$

Some important points regarding velocity potential function and stream function

(i) If stream function & velocity positional function both will exist, then flow will be both incompressible possible flow & irrotational flow.

(ii) Product of slope of Stream line & equi-potential line for two dimensional flow is -1 at all point except stagnation point i.e. Stream line & equipotential line intersect each other orthogonally at all point except stagnation point.

Let,

$m_1 =$ Slope of equipotential line

$m_2 =$ Slope of stream line

Then,

$$m_1 = \frac{-v}{u} \qquad \text{(Except stagnation point)}$$

And,

$$m_2 = \frac{u}{v} \qquad \text{(Except stagnation point)}$$

Therefore,

$$m_1 . m_2 = \frac{-v}{u} . \frac{u}{v}$$

$\Rightarrow$ $m_1.m_2 = -1$ (Except stagnation point)

(iii) If for two-dimensional steady flow u and v is given, then

(a) $\psi \rightarrow$ exist, only when u and v satisfy continuity equation

(b) $\phi \rightarrow$ exist, only when flow is irrotational i.e. $\omega_z = 0$

(c) Both ψ and ϕ will exist when u and v satisfy continuity equation and flow is irrotational i.e. $\omega_z = 0$.

(iv) If for two-dimensional steady flow ϕ is given, then

(a) $\psi \rightarrow$ exist, only when flow is possible flow and for that ϕ must satisfy Laplace equation.

(v) If for two-dimensional steady flow ψ is given, then

(a) $\phi \rightarrow$ exist, only when flow is irrotational and for that ψ must satisfy Laplace equation.

(vi) If both ψ and ϕ exist, then they satisfy the CR equation as given below:

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \text{ and } \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

Points to be Remember

Type	Existence		Satisfy Laplace Equation	
	Yes	No	Yes	No
Stream Function (ψ)	Possible flow	not possible flow	Irrotational flow	Rotational flow
Velocity Potential Function (ϕ)	Irrotational flow	Rotational flow	Possible flow	not possible flow

5.17 Flow Net

- For a fluid flow graphical representation of streamlines & equipotential lines together is known as flow net.
- Flow net exists for 2-D, steady, incompressible & irrotational flow only.

❑❑❑

6 FLUID DYNAMICS

6.1 Introduction

Fluid dynamics deals with the motion of fluid by considering cause of motion i.e. force. Major forces acting on the fluid particle are pressure force, gravity force and viscous force.

6.1.1 Euler's Equation

- Assumptions
  (i) Flow is inviscid flow
  (ii) Flow is along the stream line

$$\frac{dP}{\rho g} + \frac{ds}{g}\left(v\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t}\right) + dz = 0$$

If flow is steady:

$$\frac{dp}{\rho} + vdv + gdz = 0$$

6.1.2 Bernoulli's Equation

Assumptions:

- Flow is inviscid flow
- Flow is taking place along the streamline direction only
- Flow is steady flow
- Flow is incompressible fluid flow

$$\frac{P}{\rho g} + \frac{V^2}{2g} + z = \text{constant}$$

Pressure head = $\frac{P}{\rho g}$

Velocity Head/Dynamic Head = $\frac{V^2}{2g}$

Potential Head/ Datum Head = z

Piezometric head $h_P = \frac{P}{\rho g} + z$

Stagnation head $h_{stg} = \frac{P}{\rho g} + \frac{V^2}{2g}$

Note:

- Bernoulli's equation is applicable along the streamline only if flow is rotational whereas Bernoulli's equations is applicable across the streamlines, if the flow is irrotational.

6.2 Shortcuts for Applying Bernoulli's Equation

Case 1: Datum head is same at both points

Case 2: Velocity head is same at both points

Case 3: Pressure head is same at both points

6.2.1 Case-1: Datum Head is same at Both Points ($z_1 = z_2$)

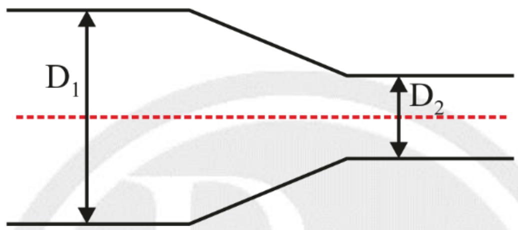

Fig.6.1 Flow across horizontal pipe

$z_2 = z_1$

$$P_1 - P_2 = \frac{1}{2}\rho(V_2^2 - V_1^2)$$

Special case:

If $D_2 = \frac{1}{2}D_1$

$$P_1 - P_2 = \frac{15}{2}\rho V_1^2$$

6.2.2 Case-2: Velocity head is same at Both Points ($V_1 = V_2$)

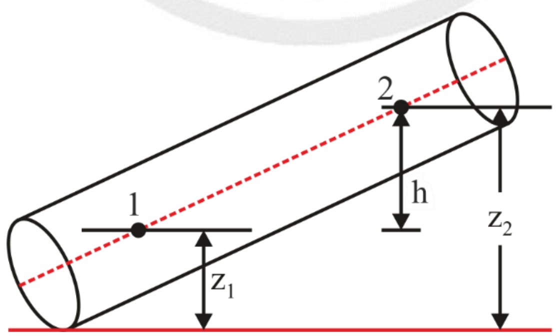

Fig.6.2 Flow across inclined pipe

$$P_1 - P_2 = \rho gh$$

6.2.3 Case-3: Pressure Head is same at Both Points ($P_1 = P_2$)

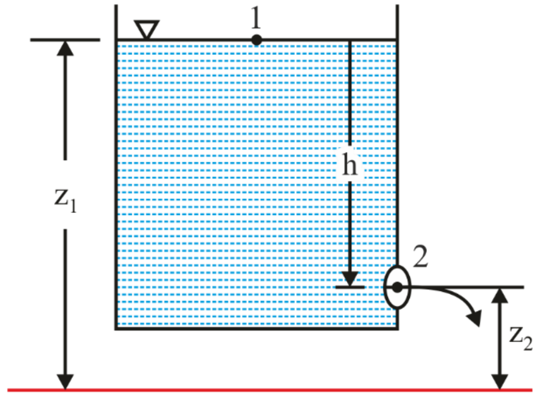

Fig.6.3 Flow through orifice

$$V_2^2 - V_1^2 = 2gh$$

If $A_1 \ggg A_2$, $V_1 \ll V_2$

, $$V_2 = \sqrt{2gh}$$

6.2.4 Modified Bernoulli's Equation

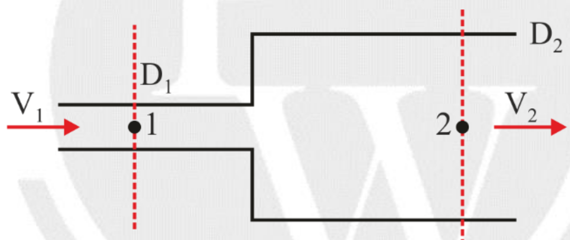

Fig.6.4 Flow across two sections in a horizontal pipe

Considering viscous flow and other losses

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + h_L$$

Where h_L = Head loss

6.3 Time Required to Empty a Rectangular Base Tank

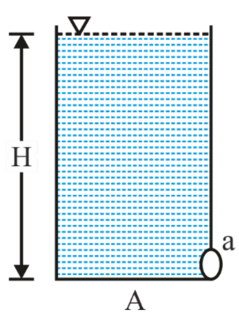

Fig. 6.5 Flow through small hole at the base

A = Tank cross sectional area
a = Tank opening area

$$T = \frac{A}{a}\sqrt{\frac{2}{g}}\sqrt{H}$$

Time required to decrease the level from H_1 to H_2

$$T = \frac{A}{a}\sqrt{\frac{2}{g}}\left[\sqrt{H_1} - \sqrt{H_2}\right]$$

6.4 Discharge Measurement with Help of Inclined Venturimeter

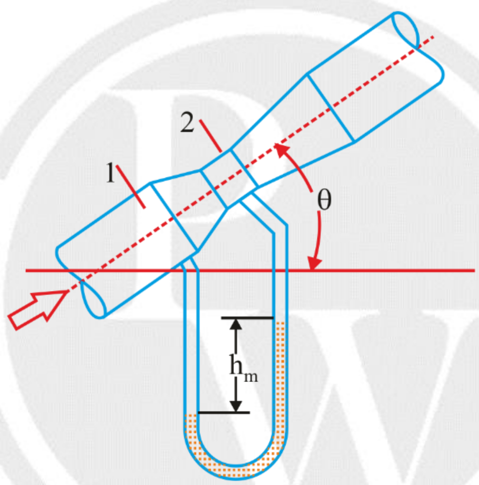

Fig. 6.6

$$Q = \left(\frac{A_1 A_2\sqrt{2g\Delta h_P}}{\sqrt{A_1^2 - A_2^2}}\right)$$

Where Δh_p = Piezometric head difference

For inclined Venturi meter $\Delta h_P = \left(\frac{P_1 - P_2}{\rho g}\right) + (z_1 - z_2)$

For horizontal Venturi meter $(z_1 = z_2)$ $\Delta h_P = \frac{P_1 - P_2}{\rho g}$

$A_1 \Rightarrow$ Cross-sectional area of pipe

$A_2 \Rightarrow$ Cross-sectional area of Throat

6.4.1 Measurement of Piezometric Head Difference with the Help of Piezometers

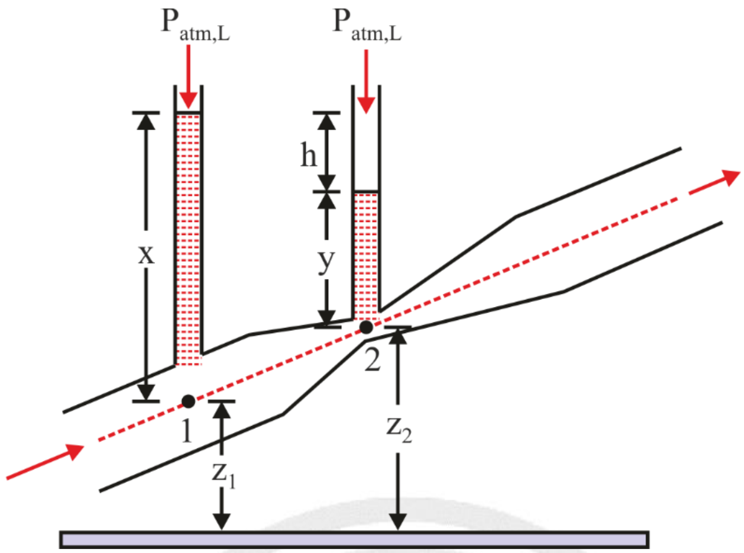

Fig. 6.7 Flow measurement through inclined Venturimeter

$$\Delta h_p = h$$

$$Q = \frac{A_1 A_2 \sqrt{2gh}}{\sqrt{A_1^2 - A_2^2}}$$

6.4.2 Measurement of Piezometric Head Difference with the Help of U-Tube Differential Manometer

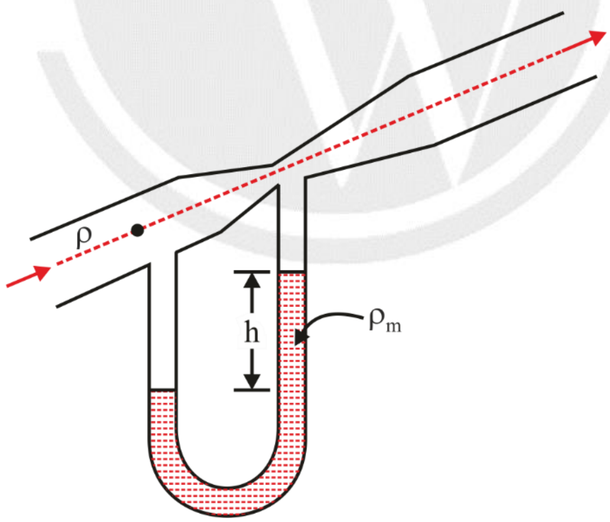

Fig.6.8 Deflection in the height of manometric column

$$\Delta h_p = \left(\frac{\rho_m - \rho}{\rho}\right) h$$

$$Q = \frac{A_1 A_2 \sqrt{2g\left(\frac{\rho_m - \rho}{\rho}\right) h}}{\sqrt{A_1^2 - A_2^2}}$$

6.4.3 Measurement of Piezometric Head Difference with the Help of Inverted U-Tube Differential Manometer $[\rho_m < \rho]$

$$\Delta h_P = \left(\frac{\rho - \rho_m}{\rho}\right) h$$

6.4.4 Expression for Actual Discharge (Q')

$$Q' = c_d Q$$

Where c_d is coefficient of discharge

h_L = head loss

$$c_d = \sqrt{1 - \frac{h_L}{\Delta h_P}}$$

$$0.95 \le c_d \le 0.99$$

6.5 Velocity Measurement for a Channel

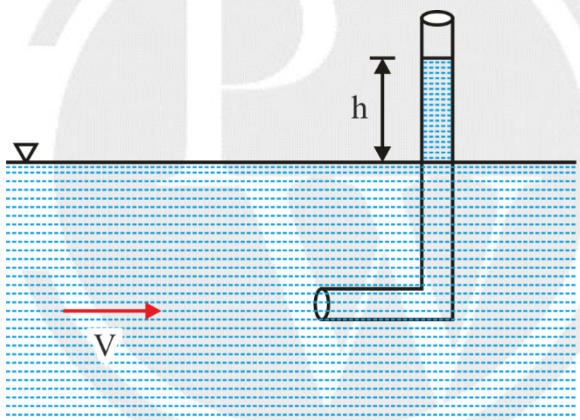

Fig.6.9 Velocity measurement through Pitot tube

$$V' = c_v\sqrt{2g\Delta h_P}\ (0.95 \le c_v \le 0.98)$$

$$\Delta h_P = h$$

$$V' = c_v\sqrt{2gh}$$

6.6 Velocity Measurement for a Pipe Flow

Case 1: with the help of piezometer and pitot tube

Case 2: with the help U - tube differential manometer & pitot tube

Case 3: with the help of Inverted U - tube differential manometer & pitot tube

6.6.1 Case 1: With the Help of Piezometer and Pitot Tube

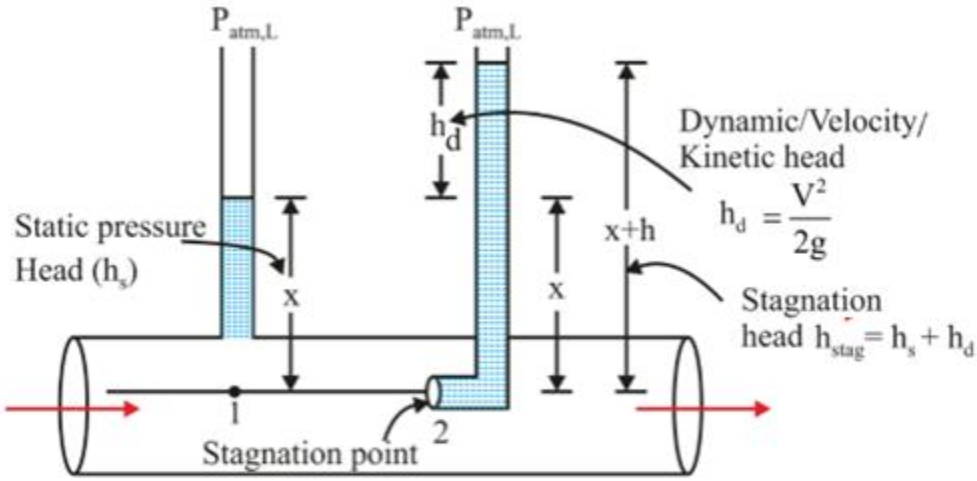

Fig.6.10 Velocity measurement with the help of piezometer & Pitot tube

$$V' = c_v\sqrt{2gh_d}$$

$$h_d = h_{stag} - h_s$$

6.6.2 Case-2: With the Help of U-Tube Differential Manometer & Pitot Tube

$$\rho_m > \rho$$

$$V' = c_V\sqrt{2g\Delta h_P}$$

$$\Delta h_P = \left(\frac{\rho_m - \rho}{\rho}\right) h$$

6.6.3 Case-3: With the Help of Inverted U-Tube Differential Manometer & Pitot Tube.

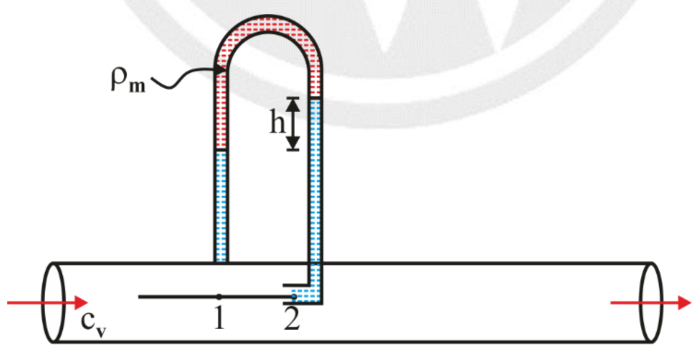

Fig.6.11 Velocity measurement with the help of inverted U-tube differential manometer and Pitot tube

$$V' = c_V\sqrt{2g\left(\frac{\rho - \rho_n}{\rho}\right) h}$$

6.7 Pitot-Static Tube

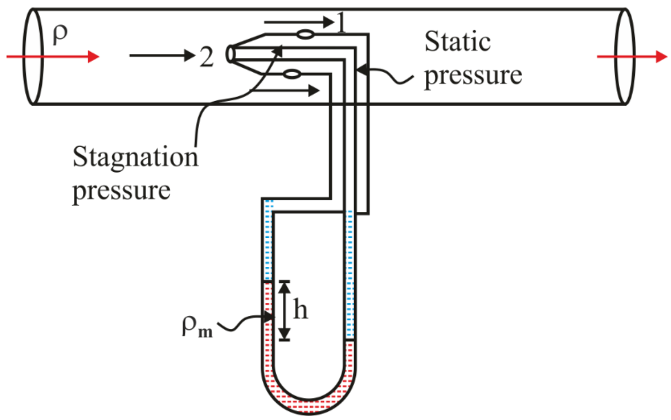

Fig.6.12 Pitot static tube

$$V' = c_v\sqrt{2g\left(\frac{\rho_m-\rho}{\rho}\right)h}$$

6.8 Force Exerted by a Flowing Fluid on a Pipe Bend

Impulse Momentum Equation

It is based on the law of conservation of momentum or on the momentum principle, which states that the net force acting on a fluid mass is equal to the change in momentum of flow per unit time in that direction.

$$F = \frac{d(mv)}{dt}$$

Where F is force acting on fluid of mass m in a short interval of time dt.

6.8.1 Force Exerted by a Flowing Fluid on a Pipe Bend

Assume:

Inviscid, steady & incompressible fluid flow and elbow is horizontal.

In the impulse momentum equation pressure is always the gauge pressure.

Various type of forces acting on the control volume -

1. Impact Force

It is the impact of fluid coming into the pipe and acts in the direction of incoming jet

2. Reaction Force of Leaving Jet

It acts due to the reaction of leaving jet, in the direction opposite to the direction of leaving jet.

3. Pressure force

Pressure force acts in the bend due to the pressure of fluid. To determine the direction of pressure force it can be assumed that the control surface is submerged in fluid and the direction of the hydrostatic force will be the direction of the pressure force.

4. Weight

Always acts downward.

5. Reaction Forces

The control surface is in equilibrium because of the reaction forces.
Consider a fluid on a pipe bend with section 1 and 2 as shown in figure. 6.12

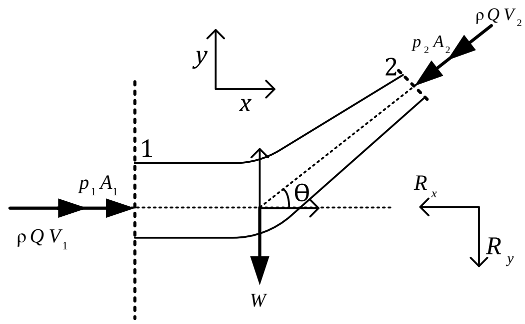

Fig.6.13 Force exerted on pipe bend

Force acting in x direction,

$$F_x = \underset{(\text{impact force})}{\rho QV_1} - \underset{(\text{reaction force})}{\rho QV_2 Cos\theta} + \underset{(\text{pressure force})}{\left(p_1 A_1 - p_2 A_2 Cos\theta\right)}$$

Force acting in y direction,

$$F_y = \underset{(\text{impulse force})}{0} - \underset{(\text{reaction force})}{\rho QV_2 \sin\theta} - \underset{(\text{pressure force})}{\left(p_2 A_2 \sin\theta\right)} - \underset{(\text{weight})}{W}$$

Reaction force by pipe bend to the fluid for equilibrium is given by,

$$R_x = -F_x$$
$$R_y = -F_y$$

6.9 Force Exerted By Jet, Striking on the Fixed Vertical Plate at the Centre

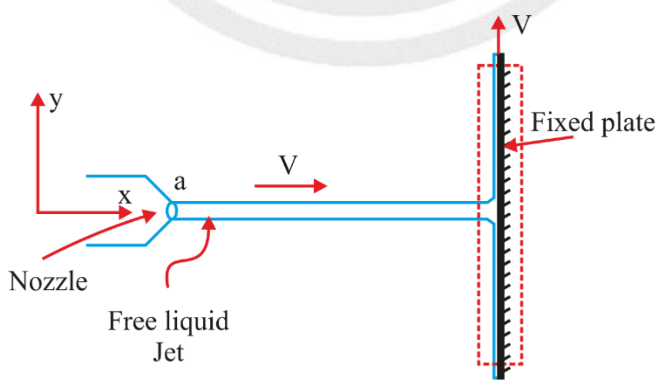

Fig.6.14 – Force Exerted by Jet, Striking on the Fixed Vertical Plate at the Centre

$$F_x = \rho a V^2 = \rho QV$$

Where Cross-sectional of jet is a and Velocity of Jet is V

$$a = \text{area of jet } \left(\frac{\pi}{4} d^2\right)$$

d = diameter of Jet / diameter of nozzle exit
Q = discharge

6.10 Angular Momentum Equation (Sprinkler Case)

Net torque = Rate of change of angular momentum

= Final angular momentum rate – Initial angular momentum rate

$$\Sigma T = (\dot{m}vr)_f - (\dot{m}vr)_i$$

6.11 Application of Angular momentum equation (sprinkler)

6.11.1 Case 1: Sprinkler is Fixed

$\Sigma T max, \omega = 0$

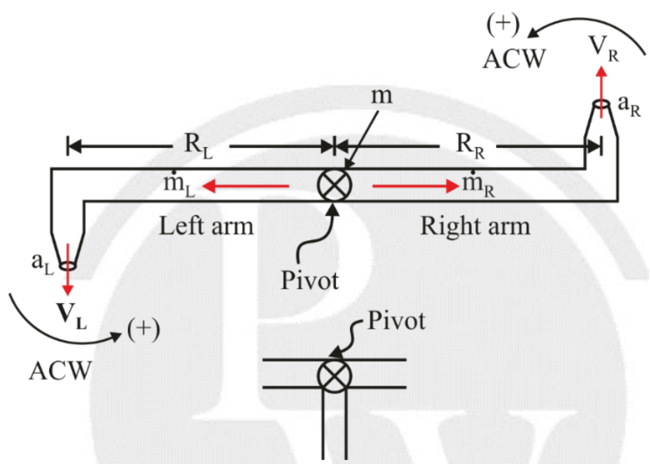

Fig.6.15 Sprinkler is fixed

Net torque acting on fluid element

$$\Sigma T = (\dot{m}_L V_L R_L) + (\dot{m}_R V_R R_R)$$

6.11.2 Case 2: Sprinkler Rotates Freely

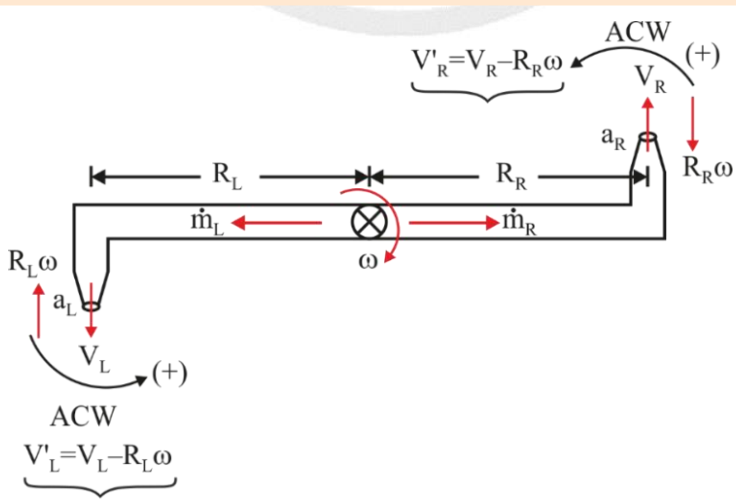

Fig.6.16 Sprinkler rotates freely

$$\dot{m}_L V_L' R_L + \dot{m}_R V_R' R_R = 0$$

6.12 Navier Stokes Equation

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}(\nabla P) + \nu\left(\nabla^2\vec{V}\right) + \frac{1}{3}\nu\left\{\nabla\left(\nabla.\vec{V}\right)\right\} + \vec{g}$$

For incompressible flow

$$\frac{D\overline{V}}{Dt} = -\frac{1}{\rho}(\nabla P) + \nu(\nabla^2\overline{V}) + g$$

For inviscid, incompressible flow $\frac{D\overline{V}}{Dt} = -\frac{1}{\rho}(\nabla P) + \overline{g}$

Where

$$\frac{D\vec{V}}{Dt} = \vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$$

$$\nabla P = \frac{\partial P}{\partial x}\hat{\imath} + \frac{\partial P}{\partial y} + \hat{\jmath} + \frac{\partial P}{\partial z}\hat{k}$$

$$(\nabla^2\vec{V}) = \nabla^2 u\hat{\imath} + \nabla^2 v\hat{\jmath} + \nabla^2 w\hat{k}$$

Here, $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}$$

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}$$

$$\left(\nabla.\vec{V}\right) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

- Flow is Incompressible Flow

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}(\nabla P) + \nu\nabla^2\vec{V} + \vec{g}$$

- Flow is Inviscid, Incompressible flow

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}(\nabla P) + \vec{g}$$

$$\frac{\partial P}{\partial x} = -\rho a_x,\ \frac{\partial P}{\partial y} = -\rho a_y,\ \frac{\partial P}{\partial z} = -\rho a_z + g$$

Note:

- For inviscid, incompressible flow

  Navier stoke' s equation in X direction $a_x = -\frac{1}{\rho}\ \frac{\partial P}{\partial x}$

  Y direction $a_y = -\frac{1}{\rho}\ \frac{\partial P}{\partial y}$

  Z direction $a_z = -\frac{1}{\rho}\ \frac{\partial P}{\partial z}$

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7 FLOW THROUGH PIPES

7.1 Loss of energy while flow through pipes

When a fluid is flowing through a pipe, the fluid experiences some resistance due to which some of the energy of fluid is lost. This loss of energy is classified as:

1. Major energy losses
2. Minor energy losses

7.2 Major energy losses

It occurs mainly due to the friction. It is calculated by the following formulae:

1. Darcy-Weisbach Formula
2. Chezy's Formula

7.2.1 Darcy-Weisbach Formula

Frictional head losses are losses due to shear stress on the pipe walls. The general equation for head loss due to friction is the Darcy-Weisbach equation, which is given below,

$$h_f = \frac{fLV^2}{2gD} = \frac{fLQ^2}{12.1d^5}$$

Where,

L = Length of pipe

D = Diameter of pipe

V = mean velocity of flow

$f = 4f'$ = friction factor

f' = coefficient of friction

h_f = Head loss due to friction

Q = Discharge

Note:

For Laminar flow

$$f_{\text{laminar}} = \frac{64}{R_e}$$

7.2.2 Chezy's Relation

Chezy's formula is given below:

$$V = C\sqrt{mi}$$

Where,

V = velocity of fluid

C = chezy's constant

$$m = \frac{A}{P} = \frac{\text{area of flow}}{\text{wetted perimeter of pipe}}$$

i = loss of head per unit length of pipe = hydraulic slope

Loss due to friction h_f is given by,

$$h_f = i.L$$

7.3 Minor energy losses

Minor losses are the losses which are minor in magnitude for very long pipes but not necessarily for shorter pipes. The minor loss of energy includes the flowing cases.

7.3.1 Sudden Expansion Loss

- Boundary layer separation leads to the formation of eddies which causes sudden expansion loss.

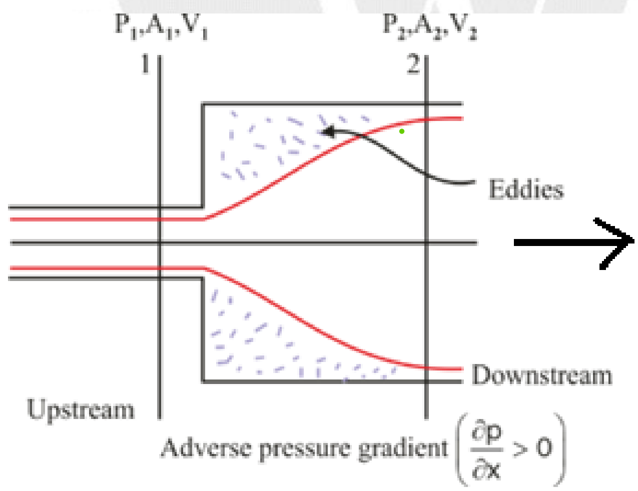


Fig.7.1 Flow representing sudden expansion

$$h_{L,SE} = \frac{(V_1 - V_2)^2}{2g} = \frac{V_1^2}{2g}\left(1 - \frac{A_1}{A_2}\right)^2$$

7.3.2 Exit Head Loss

$$h_{L,Ex} = \frac{V_1^2}{2g}$$

7.3.3 Sudden Contraction Loss

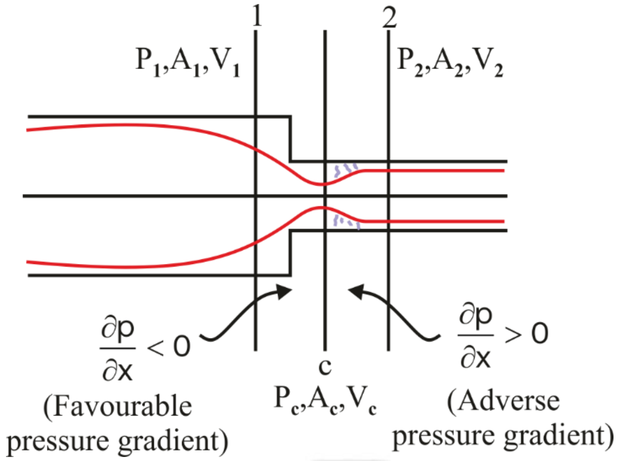


Fig.7.2 Flow representing sudden contraction

Sudden contraction loss is actually due to suddenly expansion from vena contracta to smaller diameter pipe.

$$h_{L,SC} = \left(\frac{1}{c_c} - 1\right)^2 \frac{V_2^2}{2g}$$

Where C_c = coefficient of contraction

$$\left(\frac{1}{C_c} - 1\right)^2 = k_c = \text{Contraction factor}$$

If C_c is not given then $h_{L,SC}$ is given by,

$$h_{L,SC} = \frac{0.5V_2^2}{2g}$$

7.3.4 Entrance Head Loss

$$\mathbf{h_{L,EN} = K_{L,EN}\frac{V_2^2}{2g}}$$

$K_{L,EN}$ = Entrance head loss coefficient

If $\mathbf{K_{L,EN}}$ is not given then take $\mathbf{K_{L,EN} = 0.5}$.

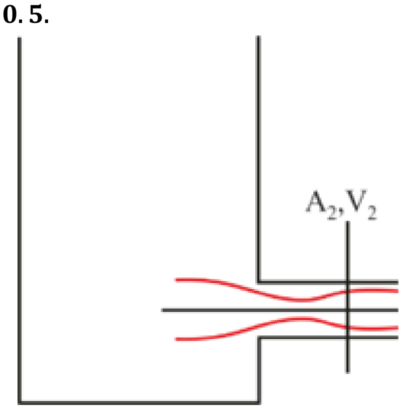


Fig.7.3 Flow representing entrance in a pipe from reservoir

7.3.5 Bend Head Loss

$$h_{L,B} = k\frac{V^2}{2g}$$

$k =$ Coefficient of Bend

7.3.6 Pipe Head Loss

$$h_{L,p} = k\frac{V^2}{2g}$$

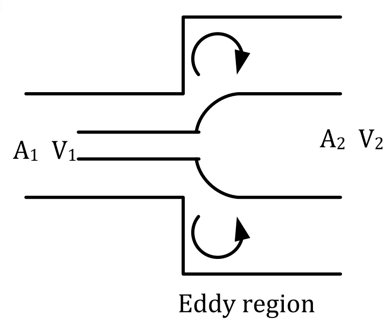
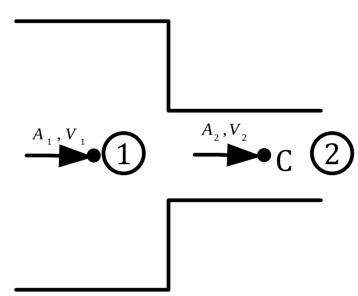
k = Coefficient of valve head loss

7.3.7 Summary

Various Losses (h_L) is calculated by using the formula,

$$h_L = K\frac{u^2}{2g}$$

Where, K is coefficient due to disturbance in pipe flow and u is reference velocity.

Conditions	**K**	**Reference Velocity(u)**	Various losses in Pipe
Major loss	$\frac{fl}{d}$	V	-
Sudden enlargement	$\left(1-\frac{A_1}{A_2}\right)^2$	V_1	
Sudden contraction	$\left(\frac{1}{C_c}-1\right)^2$ or 0.5(if C_c is not given)	V_2	

Loss at Exit	1	*V*	-
Loss at Entrance	$K_{L,EN}$	*V*	-
Loss due to bend	K(Coefficient of bend)	*V*	-
Loss due to various Pipe Fitting	K (Coefficient of pipe fitting)	*V*	-

7.4 Flow Between Two Reservoirs

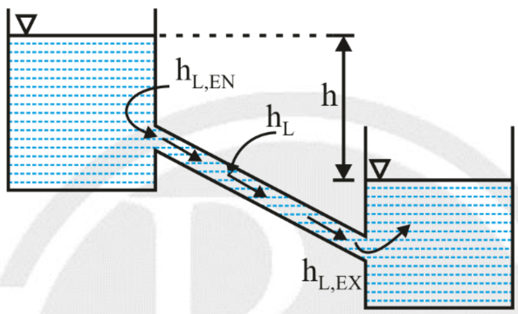


Fig.7.4 Flow between two reservoirs

$$h'_L = h_{L,EN} + h_L + h_{L,EX}$$

$$h = h_{L,EN} + h_L + h_{L,Ex}$$

$$\therefore \quad h = h'_L$$

7.5 Power Transmission Through Pipes

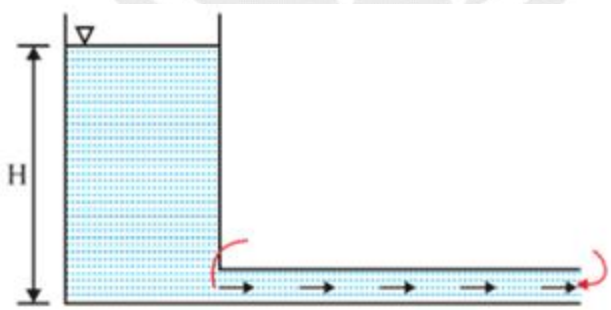


Fig.7.5 Power transmission through pipe

- Power input $P_i = \rho g Q H$
- Power output: $P_o = \rho g Q(H - h_L)$ Watt $\leftarrow$ head loss h_L in pipe
- For maximum power output

$$h_L = \frac{H}{3}$$

$\therefore$

$$(P_o)_{max} = \frac{2}{3}\rho g Q H \text{ (h-h}_L)$$

Note:

$$\eta_t = \frac{P_o}{P_i}$$

Maximum power transmission efficiency = 66.67%

7.6 Series Arrangements of Pipes

When two or more pipes of different diameters or roughness are so connected that the full discharge of the fluid from one flow into the others serially, the system represents a series pipeline.

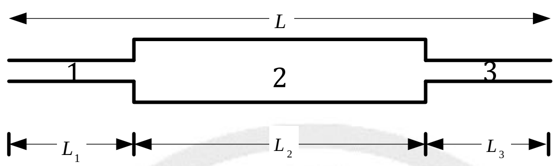


Fig.7.6 Pipes in Series

In series pipeline,

(i) Total head loss (H_L) is the sum of head loss in each Pipeline,

$$H_L = H_{L1} + H_{L2} + H_{L3}$$

Where,

H_{L1}, H_{L2}, H_{L3} are head loss in pipe 1,2 & 3 respectively

(ii) Discharge remains constants in series fitting of pipes

$$Q_1 = Q_2 = Q_3 = Q$$

- $Q_1 = Q_2 = Q$, $h_{L,T} = h_{L_1} + h_{L_2}$

7.7 Parallel Arrangement of Pipes

A combination of two or more pipes connected between two points so that the discharge divides at the first junction and rejoins at the next is known as pipes in parallel (figure 5.6). Here the head loss between the two junctions (M and N in Figure) is same for all the pipes.

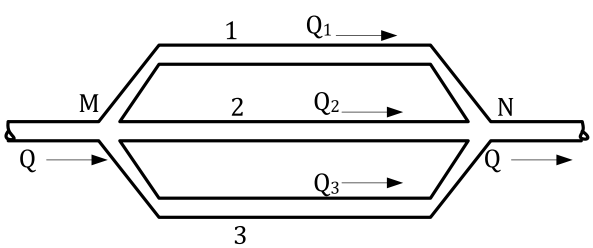


Fig.7.7 Pipes in parallel

In parallel pipeline,

(i) Head loss

$$H_{L1} = H_{L2} = H_{L3} = h_M - h_N$$

Where,

H_{L1}, H_{L2}, H_{L3} are head loss in pipe 1,2 & 3 respectively

(ii) Discharge is the sum of discharge in respective pipes

$Q = Q_1 + Q_2 + Q_3$

7.8 Equivalent Pipe

Equivalent pipe is a method of reducing a combination of pipes into a simple pipe system for easier analysis of a pipe network, such as a water distribution system.
An equivalent pipe is an imaginary pipe in which the head loss and discharge are equivalent to the head loss and discharge for the real pipe system. There are three main properties of a pipe: diameter (D), length (L), and roughness.

(i) A pipe of length L_1, diameter D_1 & friction factor f_1 will be equivalent to another pipe of corresponding parameter L_2, D_2 & f_2 if,

$$\frac{f_1 L_1}{D_1^5} = \frac{f_2 L_2}{D_2^5} \quad \ldots\text{(i)}$$

(ii) If a set of pipes described by (L_1, D_1, f_1), (L_2, D_2, f_2) ,..........are connected in series then equivalent pipe of parameter (L_e, D_e, f_e) is related as:

We know that in series total discharge, $Q = Q_e = Q_1 = Q_2 = \ldots\ldots$(ii)

And total head loss,

$$h_L = h_{L_e} = h_{L_1} + h_{L_2} + \ldots\ldots\text{(iii)}$$

Now by neglecting minor losses:

$$\Rightarrow \quad \frac{f_e L_e Q^2}{12 D_e^5} = \frac{f_1 L_1 Q_1^2}{12 D_1^5} + \frac{f_2 L_2 Q_2^2}{12 D_2^5} + \ldots \quad [\because Q = Q_e = Q_1 = Q_2 = \ldots]$$

$$\Rightarrow \quad \frac{f_e L_e Q^2}{12 D_e^5} = \frac{f_1 L_1 Q^2}{12 D_1^5} + \frac{f_2 L_2 Q^2}{12 D_2^5} + \ldots$$

$$\Rightarrow \quad \boxed{\frac{f_e L_e}{D_e^5} = \frac{f_1 L_1}{D_1^5} + \frac{f_2 L_2}{D_2^5} + \ldots\ldots} \quad \ldots\text{(iv)}$$

Where L, D, f represents Length, diameter & friction Factor respectively of corresponding pipe.

The above equation is known as Dupuit's equation.

If not specified, then assume friction factor of all the pipes and equivalent pipe same. Then equation (iv) will become,

$$\Rightarrow \quad \boxed{\frac{L_e}{D_e^5} = \frac{L_1}{D_1^5} + \frac{L_2}{D_2^5} + \ldots\ldots} \quad \ldots\text{(v)}$$

(iii) If a set of pipes described by (L_1, D_1, f_1), (L_2, D_2, f_2), ...are connected in parallel between two points then equivalent pipe of parameter (L_e, D_e, f_e) is related as:

We know that in parallel total discharge,

$$Q = Q_e = Q_1 + Q_2 + \ldots \quad \ldots\text{(vi)}$$

And total head loss,

$$h_L = h_{L_e} = h_{L_1} = h_{L_2} = \ldots \quad \ldots\text{(vii)}$$

$$\left[\because h_L = \frac{fLQ^2}{12D^5} \Rightarrow Q = \left(\frac{12D^5 h_L}{fL}\right)^{\frac{1}{2}}\right]$$

Hence equation (i) can be written as:

$$\left(\frac{12D_e^5 h_L}{f_e L_e}\right)^{\frac{1}{2}} = \left(\frac{12D_1^5 h_{L_1}}{f_1 L_1}\right)^{\frac{1}{2}} + \left(\frac{12D_2^5 h_{L_2}}{f_2 L_2}\right)^{\frac{1}{2}} + \ldots \left[\because h_L = h_{L_e} = h_{L_1} = h_{L_2} = \ldots\right]$$

$$\Rightarrow \quad \left(\frac{12D_e^5 h_L}{f_e L_e}\right)^{\frac{1}{2}} = \left(\frac{12D_1^5 h_L}{f_1 L_1}\right)^{\frac{1}{2}} + \left(\frac{12D_2^5 h_L}{f_2 L_2}\right)^{\frac{1}{2}} + \ldots$$

$$\Rightarrow \quad \boxed{\left(\frac{D_e^5}{f_e L_e}\right)^{\frac{1}{2}} = \left(\frac{D_1^5}{f_1 L_1}\right)^{\frac{1}{2}} + \left(\frac{D_2^5}{f_2 L_2}\right)^{\frac{1}{2}} + \ldots\ldots\ldots\ldots} \quad \ldots\text{(viii)}$$

Where L, D, f represents Length, diameter & friction Factor respectively of corresponding pipe.

If not specified, then assume friction factor of all the pipes and equivalent pipe same. Then equation (viii) will become,

$$\Rightarrow \quad \boxed{\left(\frac{D_e^5}{L_e}\right)^{\frac{1}{2}} = \left(\frac{D_1^5}{L_1}\right)^{\frac{1}{2}} + \left(\frac{D_2^5}{L_2}\right)^{\frac{1}{2}} + \ldots\ldots\ldots\ldots} \quad \ldots\text{(ix)}$$

Note:

Above three relations are derived by neglecting minor losses.

7.9 Syphon

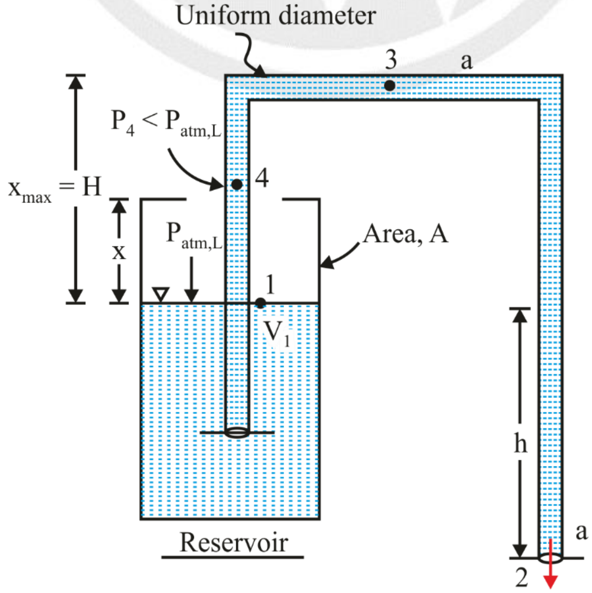


Fig.7.8 Flow through syphon

- Siphon/Syphon is a flexible pipe and Siphon effect result continuous flow
- Siphon → uniform diameter pipe
- $V_2 = \sqrt{2gh}$

  ∴ Velocity of liquid coming out of the siphon depends only on the potential head difference between free surface and end of the siphon open to atmosphere.
- $V_3 = V_2 = \sqrt{2gh}$
- The highest point of the siphon is called the SUMMIT. The pressure at the summit is less than the atmospheric pressure
- Pressure $P_4 = P_{atm,L} - \rho g(h + x)$

  ∴ Minimum pressure occurs at top/peak/summit where $x_{max} = H$
- $P_3 = P_{atm,L} - \rho g(h + H)$

7.10 Hydraulic Gradient Line and Total Energy Line

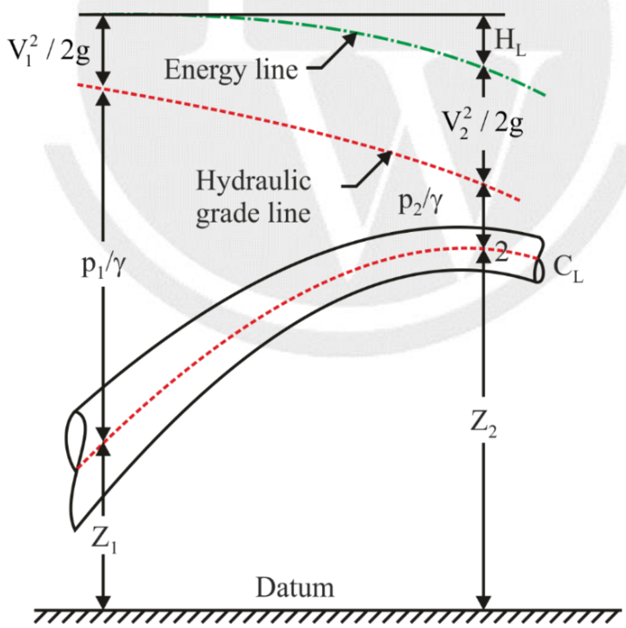


Fig.7.9 HGL & TEL for flow through pipe

- A line joining the piezometric heads at various points in a flow is known as Hydraulic Gradient Line or Piezometric Head Line (HGL).

  $h = \left(\frac{P}{\gamma} + Z\right)$
- The Total Energy Line (TEL) at any point in a flow field is

$$H = \left(\frac{P}{\gamma} + Z\right) + \frac{V^2}{2g}$$

$$H = h + \frac{V^2}{2g}$$

- The Hydraulic Gradient Line lies everywhere at a distance equal to the velocity head below the Total Energy Line.

Note:

1. TEL can never be horizontal or slope upward in the direction of flow, if the fluid is real and no energy is being added.
2. TEL and HGL are coincident and lie in the free surface for the body of liquid at rest. (e.g. Large reservoir).
3. HGL is always parallel and lower than TEL.

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8 LAMINAR FLOW

8.1 Introduction

Smooth flow of one layer of fluid over another layer is known as Laminar Flow (or) Viscous Flow.

8.1.1 Reynolds Number

It is defined as ratio of inertia force of flowing fluid and viscous force of fluid.

Inertia force $F_i = \rho A V^2$

Viscous force (F_v) = shear stress × Area

$$F_V = \tau \times A = \left(\mu . \frac{\partial u}{\partial y}\right) \times A$$

$$F_v = \mu . \frac{V}{L} \times A$$

By definition,

Reynolds Number $R_e = \frac{F_i}{F_v}$

$$R_e = \frac{\rho A V^2}{\mu . \frac{V}{L} . A}$$

$$Re = \frac{\rho V L}{\mu} = \frac{V \times L}{\mu / \rho} = \frac{V \times L}{\nu}$$

$$\Rightarrow \quad \boxed{Re = \frac{\rho V L}{\mu} = \frac{VL}{\nu}}$$

$$\frac{\mu}{\rho} = \nu = \text{kinematic viscosity}$$

In case of pipe flow, L is taken as hydraulic diameter d_H

Reynolds Number, $\boxed{R_e = \frac{\rho V d_H}{\mu} = \frac{V d_H}{\nu}}$

Where,

$$d_H = \frac{4A}{P}$$

A = Cross sectional area of pipe,

P = Perimeter of cross section

For circular pipe of diameter d,

$$d_H = \frac{4A}{P} = \frac{4 \times \frac{\pi}{4} d^2}{\pi d} = d$$

Hence Reynolds Number for circular pipe,

$$\boxed{R_e = \frac{\rho V d}{\mu} = \frac{Vd}{\nu}}$$

Critical Reynolds Number

The Reynolds number value below which the flow can be definitely considered to the laminar is known as Critical Reynolds number.

The value depends upon the flow conditions and the geometry of the flow. For circular conduits a value of Recrit = 2000 is generally taken as the Critical Reynolds number.

8.1.2 Fully Developed Flow

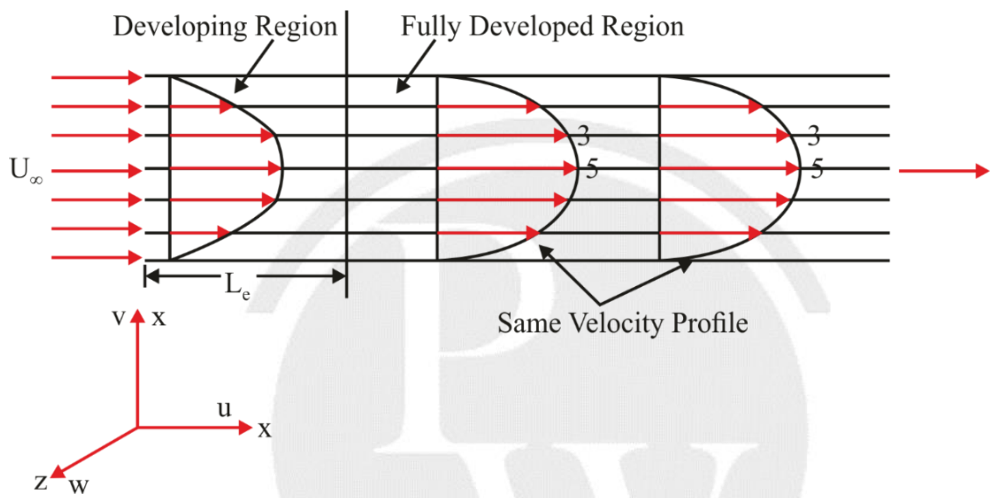

Fig. 8.1 Fully developed flow through pipe

- In fully developed flow each fluid particle moves at constant velocity along a stream line and velocity profile remains unchanged in flow direction.
- There is no motion in radial direction. Hence component of velocity in the direction normal to the axis is zero everywhere.

$$u \neq 0, \frac{\partial u}{\partial x} = 0$$

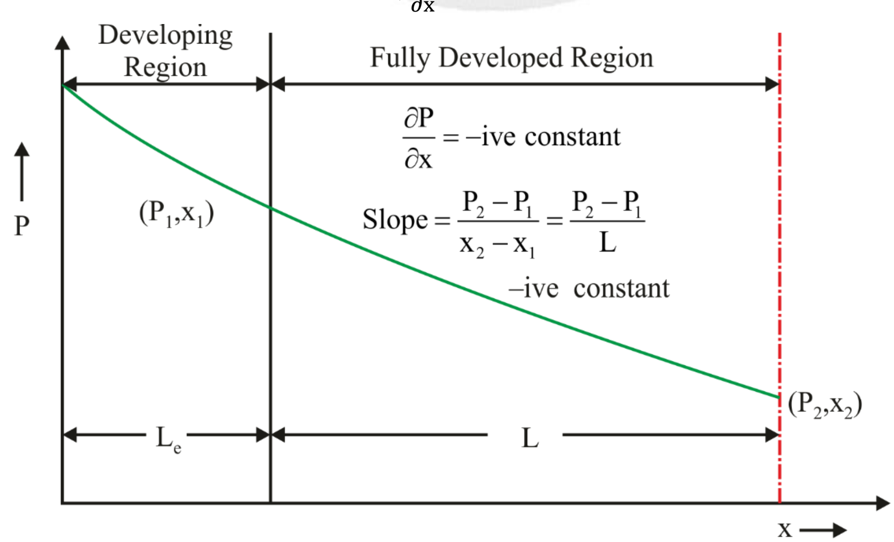

Fig. 8.2 Variation of pressure gradient in developing and fully developed flow

- For steady and incompressible fluid flow, average velocity in developing flow region and fully developed flow region are identical and Re is also same at every cross section.

8.3 Laminar Flow Through Circular Pipe (Hagen Poiseuille Flow)

- Steady, uniform
- Incompressible
- Fully Developed Flow

1. Shear Stress (τ) Distribution

$$\Rightarrow \quad \boxed{\tau = \left(-\frac{\partial p}{\partial x}\right) \cdot \frac{r}{2}} \quad \ldots\text{(i)}$$

The negative sign on $\frac{\partial p}{\partial x}$ indicates decrease in pressure in the direction of flow.

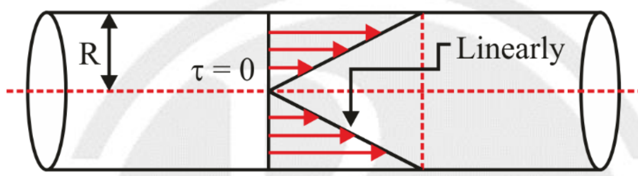

Fig. 8.3 Shear stress variation in flow through pipe

- At center line r = 0 then τ = 0
- At wall r = R then

$$\tau_w = \left(-\frac{dP}{dx}\right)\left(\frac{R}{2}\right)$$

2. Velocity Distribution

$$\boxed{u = -\frac{1}{4\mu}\frac{\partial p}{\partial x}\left[R^2 - r^2\right]} \quad \ldots\text{(ii)}$$

3. Maximum velocity (u_{max})

Velocity u will be maximum at $r = 0$, hence putting $r = 0$ in equation (ii) will get maximum velocity,

$$\boxed{u_{max} = \frac{1}{4\mu}\left(-\frac{\partial p}{\partial x}\right)R^2} \quad \ldots\text{(iii)}$$

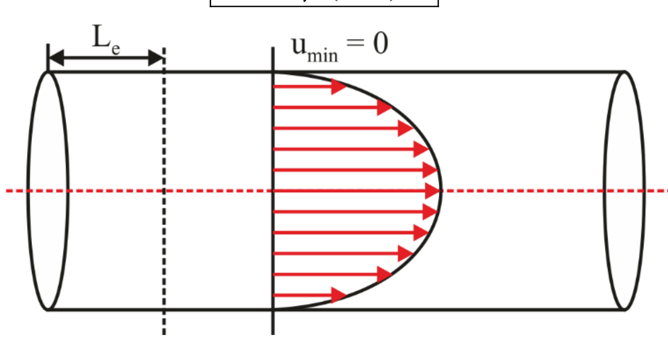

Fig.8.4 Velocity distribution for flow through pipe

4. Average Velocity (u_{avg})

$$\Rightarrow \qquad \boxed{u_{avg} = \frac{u_{max}}{2}}$$

5. Location of Average Velocity:

$$\Rightarrow \qquad \boxed{r = \frac{R}{\sqrt{2}} = 0.707R}$$

6. Discharge (Q) Through A Pipe

$$\boxed{Q = u_{avg} \times A = \frac{\pi}{8\mu}\left(\frac{-\partial p}{\partial x}\right)R^4}$$

7. Pressure drop ($p_1 - p_2$) between 1 & 2

$$\Rightarrow \qquad \boxed{p_1 - p_2 = \frac{8\mu u_{avg} L}{R^2}}$$

8. Drop in pressure head (h_f) between 1 and 2

$$h_f = \frac{p_1 - p_2}{\rho g}$$

$$\Rightarrow \qquad \boxed{h_f = \frac{8\mu u_{avg} L}{\rho g R^2}} \qquad \left[\because R = \frac{D}{2}\right]$$

$$\Rightarrow \qquad h_f = \frac{32\mu u_{avg} L}{\rho g D^2} \qquad \left[\because u_{avg} = \frac{Q}{A} = \frac{Q}{\frac{\pi}{4}d^2}\right]$$

$$\Rightarrow \qquad \boxed{h_f = \frac{128\mu Q L}{\rho g \pi D^4}}$$

Note:

Hence for laminar flow through circular pipe having discharge Q,

loss of head $h_f \propto \frac{1}{D^4}$

9. Power (P) Required to Maintain the Laminar Flow

$$\boxed{P = (p_1 - p_2) \times Q}$$

Note:

In case of inclined plate gravity force will also active, therefore pressure (p) will be replaced by (p + ρgz) i.e. use $d(p + \rho gz)$ in place of dp. Then

$$-\frac{\partial p}{\partial x} \rightarrow -\frac{\partial(p+\rho gz)}{\partial x}$$

and, $h_f = \frac{p_1 - p_2}{\gamma}$ will change to

$$h_f = \frac{(p_1+\gamma z_1)-(p_2+\gamma z_2)}{\gamma} = \left(\frac{p_1}{\gamma}+z_1\right)-\left(\frac{p_2}{\gamma}+z_2\right)$$

8.4 Friction Factor

- For steady incompressible Newtonian fluid, fully developed laminar flow through pipe.

$$f = \frac{64}{Re}$$

- Minimum value of Friction factor for laminar flow through pipe.

$$F_{min} \cong 0.032$$

- For laminar flow

$$(Re)_{max} \cong 2000$$

8.5 Momentum Correction Factor

- It is the ratio of the momentum rate based on V_{act} to the momentum rate based on V_{avg}

$$\beta = \frac{\dot{P}_{act}}{\dot{P}_{avg}}$$

$$\beta = \frac{1}{A}\int\left(\frac{u}{u_{av}}\right)^2 dA$$

- For flow through pipe $u = u(r)$ only then

$$\beta = \frac{1}{\pi r^2}\int\left(\frac{u}{u_{av}}\right)^2 2\pi r dr$$

$$\beta = \frac{4}{3} \quad \text{(Hagen Poiseuille Flow)}$$

8.6 Kinetic Energy Correction Factor

- Ratio of kinetic energy rate based on actual velocity to the KE rate based on Average velocity
- Flow through pipe u = u(r) only,

$$\alpha = \frac{1}{A}\int\left(\frac{u}{u_{av}}\right)^3 dA$$

$\alpha = 2$ (Hagen Poiseuille Flow)

- According to Bernoulli's Equation, for steady incompressible Newtonian fluid, fully developed laminar flow through circular pipe.

$$\frac{P_1}{\rho g} + \alpha\frac{V_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \alpha\frac{V_2^2}{2g} + z_2$$

Note:

For uniform velocity β = 1 and α = 1

8.7 LAMINAR FLOW BETWEEN TWO STATIONARY PARALLEL PLATES

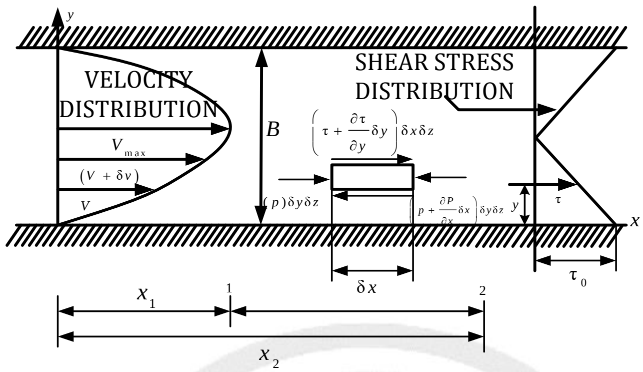

Fig.8.4

Since both the plate is fixed, therefore,

At $y = 0, u = 0$

& at $y = B, u = 0$

Pressure gradient is given by,

$$\frac{\partial p}{\partial x} = \frac{d\tau}{dy} \qquad \left[\because \tau = \mu \frac{du}{dy}\right]$$

$$\frac{\partial p}{\partial x} = \mu \frac{d^2u}{dy^2}$$

1. Velocity Distribution

$$\boxed{u = \left(-\frac{1}{2\mu}\frac{dp}{dx}\right)\left(By - y^2\right)}$$

Above equation is parabolic, hence velocity profile is parabolic in this case.

and $$\frac{du}{dy} = \left(-\frac{1}{2\mu}\frac{dp}{dx}\right)\left(B - 2y\right)$$

2. Maximum velocity (u_m)

Point of maximum velocity,

$$\Rightarrow \qquad \boxed{y = \frac{B}{2}}$$

Maximum Velocity,

$$\boxed{u_m = \left(-\frac{1}{8\mu}\frac{\partial p}{\partial x}\right)B^2}$$

3. Average Velocity (u_{avg})

$$\boxed{u_{avg} = \frac{2}{3}u_m}$$

4. Discharge per unit width (Q')

$$\Rightarrow \quad \boxed{Q = u_{av} \times B = \frac{B^3}{12\mu}\left(-\frac{\partial p}{\partial x}\right)}$$

5. Pressure drop ($p_1 - p$) between 1 & 2

$$\left(-\frac{\partial p}{\partial x}\right) = \frac{12\mu u_{avg}}{B^2}$$

$$\Rightarrow \quad \frac{p_1 - p_2}{L} = \frac{12\mu u_{avg}}{B^2}$$

$$\Rightarrow \quad \boxed{p_1 - p_2 = \frac{12\mu u_{avg} L}{B^2}} \quad \text{...(x)}$$

6. Drop in pressure head (h_f) between 1 and 2

$$h_f = \frac{p_1 - p_2}{\rho g}$$

$$\Rightarrow \quad \boxed{h_f = \frac{12\mu u_{avg} L}{\rho g B^2}} \quad \text{...(xi)}$$

7. Power (P) Required to Maintain the Laminar Flow

$$P = (p_1 - p_2) \times Q$$

8. Shear Stress (τ) Distribution

$$\tau = \mu\frac{du}{dy}$$

$$\tau = \left(-\frac{\partial p}{\partial x}\right)\left(\frac{B}{2} - y\right)$$

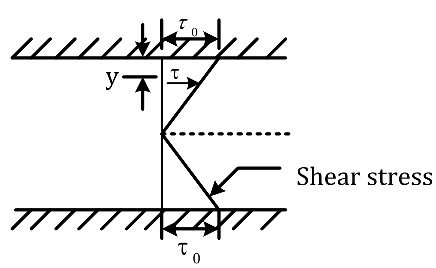

Fig.8.5

Note:

In case of inclined plate gravity force will also active, therefore pressure (p) will be replaced by (p + ρgz) i.e. use $d(p + \rho gz)$ in place of dp. Then

$$-\frac{\partial p}{\partial x} \rightarrow -\frac{\partial (p+\rho gz)}{\partial x}$$

and, $$h_f = \frac{p_1 - p_2}{\gamma} \text{ will change to}$$

$$h_f = \frac{(p_1+\gamma z_1)-(p_2+\gamma z_2)}{\gamma} = \left(\frac{p_1}{\gamma}+z_1\right)-\left(\frac{p_2}{\gamma}+z_2\right)$$

8.8 Flow Between Two Plate – One Moving & Other Fixed

The flow between two parallel plates where one plate is moving & other is fixed is known as Coutte Flow as shown in figure 7.7. given below:

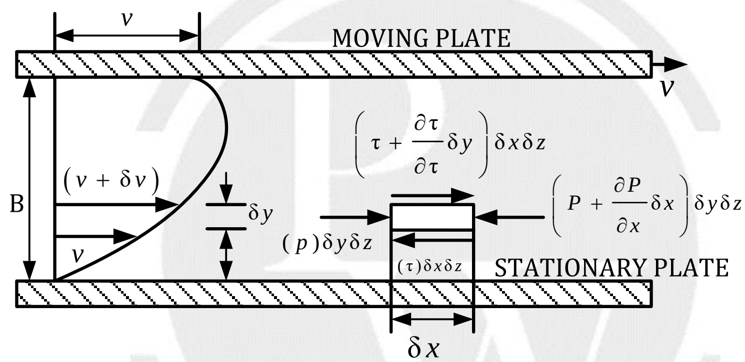

Fig.8.6

The velocity distribution for Couette Flow is given by,

$$u = \frac{1}{\mu}\left(\frac{\partial p}{\partial x}\right)\frac{y^2}{2} + C_1 y + C_2$$

Applying Boundary condition ($u = 0\, at\ y = 0$ & $u = V\ at\ y = B$) in above equation,

$$u = \frac{Vy}{B} - \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(By - y^2\right)$$

If pressure gradient in the direction of flow is zero, then,

$$u = \frac{Vy}{B} \leftarrow \text{linear}$$

The above equation is linear. This is the particular case of Coutte flow which is known as Plain Couette Flow or simply shear flow.

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9 TURBULENT FLOW

9.1 Introduction

9.1.1 Turbulent Flow

- In Turbulent flow, fluid particles move in disorderly manner causing the formation of swirling eddies.
- As a result, Turbulent flow is associated with much higher values of friction, heat & mass transfer coefficients in Turbulent flow, we have larger loss of energy and therefore a greater drop in pressure.

9.1.2 Reynolds Decomposition Principle to specify the velocity field

$$u = \bar{u} + u'$$

Where u = Instantaneous velocity in $x-$ direction

$\bar{u}$ = Time average velocity $x-$ direction

u' = Fluctuations in x-direction (it can be + ive / -ive / 0)

$$\bar{u}' = 0, \bar{u}'^2 \neq 0$$

Note:
$\bar{u}$ will not change with time.

9.1.3 Turbulence Intensity (I_t)

$$I_t = \frac{\sqrt{\overline{u'}^2}}{\bar{u}}$$

9.2 SHEAR STRESS IN TURBULENT FLOW

9.2.1 As Per Boussinesq's Theory

$\tau = \tau_v + \tau_t$

$\tau = \mu \frac{d\bar{v}}{dy} + \eta \frac{d\bar{v}}{dy}$

Where,

τ_v =Shear stress due to viscosity

τ_t =Shear stress due to Turbulence

μ = dynamic coefficient of viscosity (fluid characteristic),

$\bar{v}$ = Time average of velocity at the point of consideration

η = eddy viscosity coefficient (flow characteristic)

η =0 for laminar flow.

Eddy viscosity comes in picture due to the turbulence effect.

9.2.2 As per Reynolds Theory

$$\tau = \mu \frac{du}{dy} - \rho \overline{u'v'}$$

Where,

μ = dynamic coefficient of viscosity (fluid characteristic)

$\overline{u'v'}$ = Time average of product of fluctuating velocity u'&v'

9.2.3 As per Prandtl's mixing Length Theory

$$\tau = \mu \frac{d\bar{v}}{dy} + \rho l^2 \left(\frac{d\bar{v}}{dy}\right)^2$$

If viscous effect is neglected,

$$\tau = \rho l^2 \left(\frac{d\bar{v}}{dy}\right)^2$$

Where,

μ = dynamic coefficient of viscosity (fluid characteristic),

$\bar{v}$ = Time average of velocity at the point of consideration

l = mixing length

9.3 VARIOUS LAYERS IN TURBULENT FLOW

Turbulent flow along a wall can be considered to consist of four regions, characterized by the distance from the wall.

(i) Laminar Viscous Sublayer:

The very thin layer next to the wall where viscous effects are dominant is the viscous sublayer. The velocity profile in this layer is very nearly linear, and the flow is streamlined.

(ii) Buffer layer:

Next to the viscous sublayer is the buffer layer, in which turbulent effects are becoming significant, but the flow is still dominated by viscous effects.

(iii) Overlap layer:

Above the buffer layer is the overlap (or transition) layer, also called the inertial sublayer, in which the turbulent effects are much more significant, but still not dominant.

(iv) Outer layer:

Above the overlap layer is the outer (or turbulent) layer in the remaining part of the flow in which turbulent effects dominate over molecular diffusion (viscous) effects.

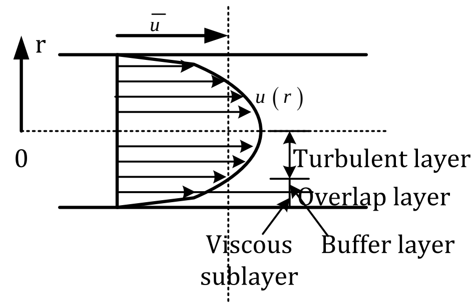

Fig.9.1

9.4 Hydro Dynamically Smooth and Rough Boundaries

If the average height of irregularities (k) is greater than the thickness of laminar sub-layer (δ') then the boundary is called hydro dynamically smooth Boundaries.

If the average height of irregularities (k) is less than the thickness of laminar sub-layer (δ') then the boundary is called hydro dynamically rough Boundaries. In general it can be understood from the following figure 9.2 and 9.3.

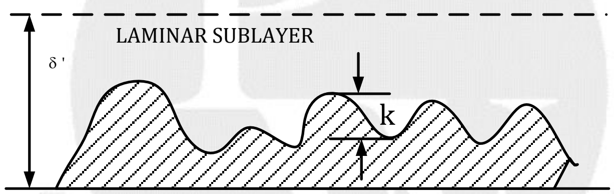

Fig.9.1 Absolute Smooth boundary

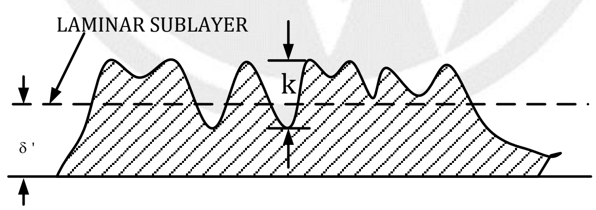

Fig.9.3 Rough boundary

On the basis of NIKURADSES EXPERIMENT the boundary is classified as.

Hydro dynamically smooth Boundaries $\frac{k}{\delta'} < 0.25$

Boundary in transition: $0.25 < \frac{k}{\delta'} < 6.0$

Hydro dynamically Rough Boundaries $\frac{k}{\delta'} > 6.0$

Expression of thickness of laminar sublayer,

$$\delta' = \frac{11.6\upsilon}{u_f} \quad \ldots\text{(i)}$$

Roughness Reynolds Number

It is given by,

$$\frac{u_f k}{\upsilon} \quad \ldots\text{(ii)}$$

Specific Roughness

It is given by,

$$\frac{R}{k} \quad \ldots\text{(iii)}$$

Where,

$u_f = \sqrt{\frac{\tau_0}{\rho}}$ = Shear or friction velocity

$k =$ average height of roughness

υ = kinematic viscosity.

$R =$ radius of the pipe

Rewriting Equation (i)

$$\delta' = \frac{11.6\upsilon}{u_f}$$

$$\Rightarrow \quad \frac{k}{\delta'} = \frac{u_f k}{11.6\upsilon}$$

$$\Rightarrow \quad \frac{u_f k}{\upsilon} = 11.6\frac{k}{\delta'}$$

Now, by using above relation, in terms of roughness Reynolds number $\left(\frac{u_* k}{\upsilon}\right)$ the boundary is classified as:

Hydro dynamically smooth Boundaries $\frac{u_* k}{\upsilon} < 3$

Boundary in transition: $3 < \frac{u_* k}{\upsilon} < 70$

Hydro dynamically Rough Boundaries $\frac{u_* k}{\upsilon} > 70$

9.5 VELOCITY DISTRIBUTION FOR TURBULENT FLOW IN PIPES

Prandtl's velocity distribution equation is given by,

$$u = u_{max} + 2.5u_* \log_e\left(\frac{y}{R}\right)$$

Where

$u_* = \sqrt{\frac{\tau_0}{\rho}}$ = Shear or friction velocity

$y =$ distance from pipe wall,

ρ = Density of fluid

u_{max} = Centerline velocity

R = Radius of pipe

Note

The above equation is valid for both smooth and rough pipe boundaries.

9.5.1 Local Velocity

For turbulant flow in a pipe the velocity distribution at a distance y from boundary is given by,

(i) For Hydro Dynamically Smooth Boundaries

$$\frac{u}{u_*} = 5.75\log_{10}\left(\frac{u_* y}{\upsilon}\right) + 5.5$$

(ii) For Hydro Dynamically Rough Boundaries

$$\frac{u}{u_*} = 5.75\log_{10}\left(\frac{y}{k}\right) + 8.5$$

Where,

$u =$ Local velocity at distance y from boundary

$u_* = \sqrt{\frac{\tau_0}{\rho}}$ = Shear or friction velocity

υ = kinematic viscosity

$k =$ average height of roughness

9.5.2 Velocity Distribution in Terms of Mean Velocity

The mean velocity (V) in a pipe of radius R is obtained by integrating the above equation over the entire area of pipe and dividing it by cross sectional area of pipe (i.e. πR^2).

This gives,

(i) For Hydro Dynamically Smooth Boundaries

$$\frac{V}{u_*} = 5.75\log_{10}\left(\frac{u_* R}{\upsilon}\right) + 1.75$$

(ii) For Hydro Dynamically Rough Boundaries

$$\frac{V}{u_*} = 5.75 \log_{10}\left(\frac{R}{k}\right) + 4.75$$

Where,

V = Mean velocity

$u_* = \sqrt{\frac{\tau_0}{\rho}}$ = Shear or friction velocity

y = distance from pipe wall,

υ = kinematic viscosity

k = average height of roughness

R = Radius of pipe

Difference in Local velocity (u) & Mean velocity (V) is given by,

$$\frac{u-V}{u_*} = 5.75 \log_{10}\left(\frac{y}{R}\right) + 3.75 \text{ (valid for Both smooth and rough pipes)}$$

9.6 Friction Factor

Friction Factor '*f*' for Laminar Flow

$f = \frac{64}{R_e}$ where Re = Reynolds number

Friction Factor (*f*) For Turbulent Flow in Smooth Pipes

1. Blasius Equation

$$f = \frac{0.316}{(R_e)^{1/4}} \quad (\text{for } 4\times10^3 < \text{Re} < 10^5)$$

$$\frac{1}{\sqrt{f}} = 2.0 \log_{10}\left(\text{Re}\sqrt{f}\right) - 0.8 (\text{for } 5\times10^4 < \text{Re} < 4\times10^7)$$

Friction Factor (f) For Turbulent Flow in Rough Pipes.

$$\frac{1}{\sqrt{f}} = 2.0 \log_{10}\left(\frac{R}{k}\right) + 1.74$$

Friction Factor for Commercial Pipes

$$\frac{1}{\sqrt{f}} - 2.0 \log_{10}\left(\frac{R}{k}\right) = 1.74 - 2.0 \log\left(1 + 18.7 \frac{R/k}{R_e\sqrt{f}}\right)$$

Where,

f = Friction factor

$y =$ Distance from pipe wall,

R_e = Reynolds number

R = Radius of pipe

$k =$ Average height of roughness

Note:

The above equation shows that for smooth pipe friction factor depends upon Reynolds number (R_e) only and for rough pipe it depends upon Reynolds number (R_e) and Relative smoothness ($\frac{R}{k}$) both.

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10 BOUNDARY LAYER THEORY

10.1 Introduction

When a real fluid flows past a solid body or a solid wall, the fluid particles adhere to the boundary and condition of no slip occurs. This means that the velocity of fluid close to the wall will be same as that of the wall. If the wall is stationary, the velocity of fluid at the wall will be zero. Farther away from the wall, the velocity will be higher and as a result of this variation of velocity, the velocity gradient $\frac{du}{dy}$ will exist. The velocity of fluid increases from zero velocity on the stationary boundary to free-stream velocity (U) of the fluid in the direction normal to the boundary. This variation of velocity from zero to free-stream velocity in the direction normal to the boundary takes place in a narrow region in the vicinity of solid boundary. This narrow region of the fluid is called boundary layer. The theory dealing with boundary layer flows is called boundary layer theory. In this region velocity gradient $\frac{du}{dy}$ exists and hence the fluid exerts a shear stress on the wall in the direction of motion.

The velocity in the remaining fluid, which is outside the boundary layer is constant and equal to free-stream velocity. As there is no variation of velocity in this region, the velocity gradient $\frac{du}{dy}$ becomes zero. As a result of this the shear stress is zero.

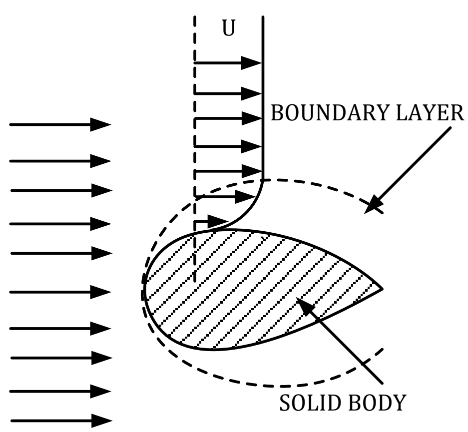

Fig.10.1

10.2 Boundary Layer Thickness

Boundary layer thickness is the distance from solid surface in normal direction at which velocity becomes equal to the 99% of free stream velocity.

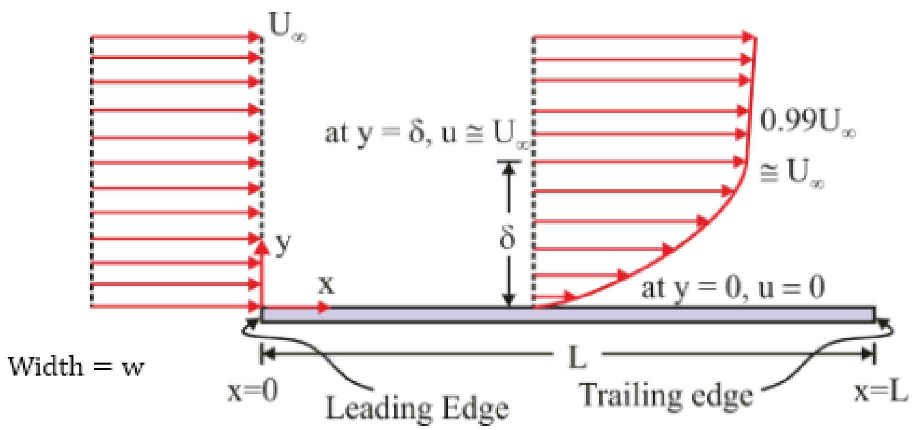

Fig.10.2 Flow over flat plate

10.2.1 Boundary Conditions

At – $x = 0 \Rightarrow \delta = 0$, $\;y = 0 \Rightarrow u = 0$, $\;y = \delta \Rightarrow u \cong U_\infty$, $\;y \geq \delta \Rightarrow \frac{du}{dy} = 0$

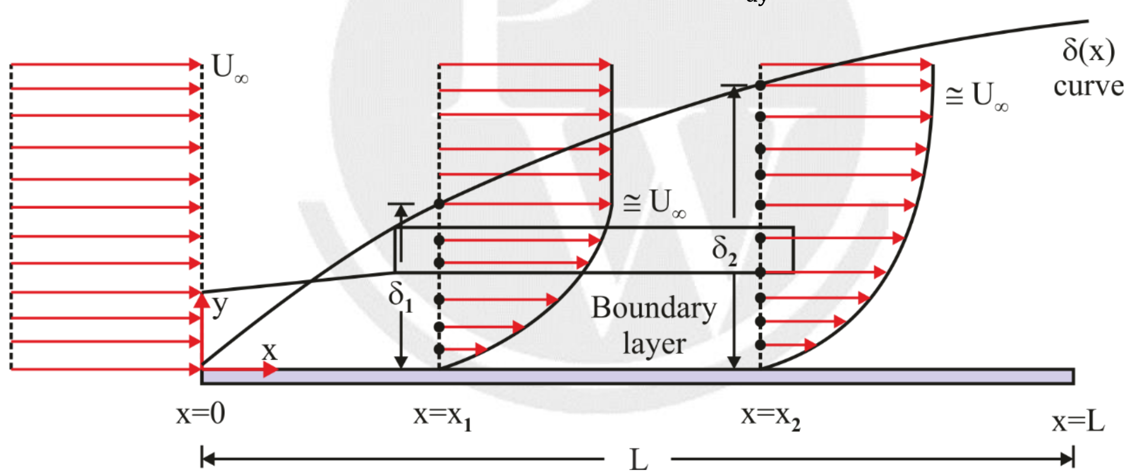

Fig.10.3 Boundary condition for flow over flat plate

Note:

As the flow moves downstream, retarded fluid particles further retard and they retard other fluid particles also, leading to growth of boundary layer. Hence, the thickness of boundary layer keeps on increasing in downstream direction.

10.3 Reynolds Number for Flow Over Plate

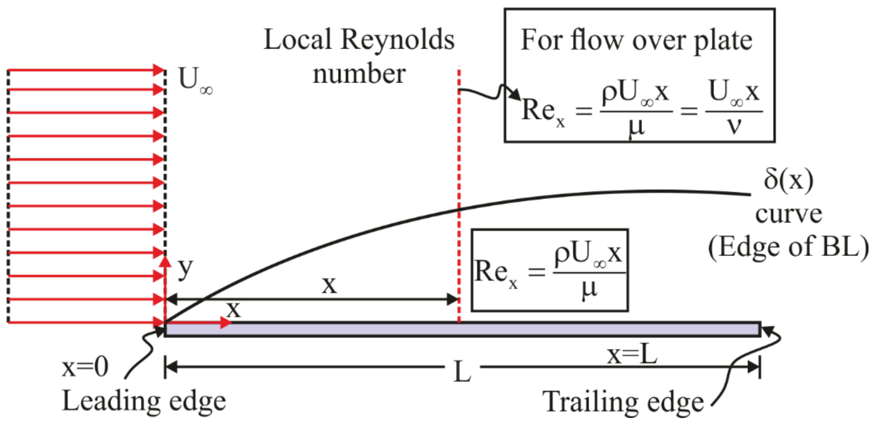

Fig. 10.4 Reynolds number for flow over flat plate

$$Re = \frac{\rho VL}{\mu}$$

$$\mathbf{Re_x = \frac{\rho U_0 x}{\mu}}$$

10.4 Development of a Boundary Layer Over Flat Plate

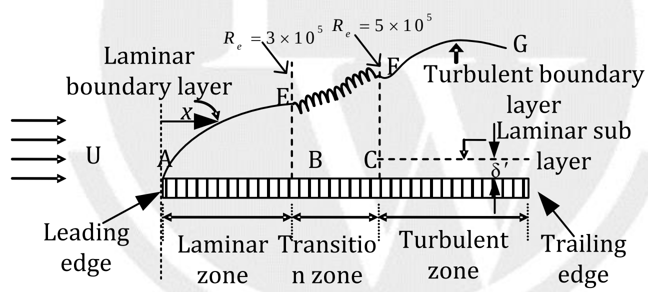

Fig.10.5 Development of boundary layer over flat plate

Near the leading edge the flow is laminar irrespective of whether the flow of incoming streams Laminar (or) Turbulent.

10.4.1 Laminar Boundary Layer

This is defined as the boundary up to which flow is laminar. This is shown by distance AB in figure. 10.5. In a laminar boundary layer any exchange of mass or momentum takes place only between adjacent layers on a microscopic scale which is not visible to the eye. The velocity distribution in Laminar flow is Parabolic.

In figure 10.5 flow from A to B is completely laminar. The Reynold's No. (R_e) at point B is 3×10^5.

In laminar boundary layer, boundary layer thickness (δ) is given by,

$$\delta = \frac{5x}{\sqrt{R_{e_x}}} \qquad ...(i)$$

Reynold's no. (R_{e_x}) is given by,

$$\left(R_{e_x}\right) = \frac{U\times x}{\upsilon}$$

Where,

$U =$ Free stream velocity

$x =$ Distance from leading edge

$\upsilon =$ Kinematic Viscosity

From above two equations we can conclude that for laminar boundary layer,

$\delta \propto \sqrt{x}$

10.4.2 Turbulent Boundary Layer

If length of plate is more than the distance, x calculated from (i), then the laminar boundary layer becomes unstable and there is transition of motion of fluid from laminar to turbulent boundary.
The velocity distribution in turbulent flow is logarithmic.

The short length over which the boundary flow changes from laminar to turbulent is called transition zone. This is shown by distance BC in figure 10.5.

At point C where Reynold's no. (R_e) is 5×10^5 . transition completes & flow becomes completely turbulent.

A turbulent boundary layer on the other hand is marked by mixing across several layers of it. The mixing is now on a macroscopic scale. Packets of fluid may be seen moving across. Thus, there is an exchange of mass, momentum and energy on a much bigger scale compared to a laminar boundary layer. A turbulent boundary layer forms only at larger Reynolds numbers.

In turbulent boundary layer, boundary layer thickness (δ) is given by,

$$\delta = \frac{0.377x}{\left(R_{e_x}\right)^{\frac{1}{5}}}$$

Reynold's no. (R_{e_x}) is given by

$$\left(R_{e_x}\right) = \frac{U \times x}{\upsilon}$$

Where,

$U =$ Free stream velocity

$x =$ Distance from leading edge

$\upsilon =$ kinematic viscosity

From above two equations we can conclude that for laminar boundary layer,

$\delta \propto x^{\frac{4}{5}}$

10.4.3 Laminar Sub-Layer

In the vicinity of the wall t in turbulent boundary layer there is very thin layer of the fluid where turbulent fluctuations are damped. In this region the velocity variation is influenced only by viscous effect. This region is called laminar sub-layer. Though the velocity distribution would be parabolic in laminar sub-layer, but in view of very small thickness we can assume that velocity distribution is linear in this region, thus velocity gradient can be considered constant. Therefore, shear stress in laminar sub-layer equal to constant & will be equal to boundary shear stress (τ_0). In figure 10.5-line DD' is laminar sub-layer, which is given by,

$$\tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu\frac{u}{y}$$

10.5 Various thickness involved in Boundary Layer Theory

10.5.1 Displacement Thickness

It is defined as the distance, measured perpendicular to the boundary of the solid body, by which the boundary should be displaced to compensate for the reduction in flow rate due to velocity defect on account of boundary layer formation.

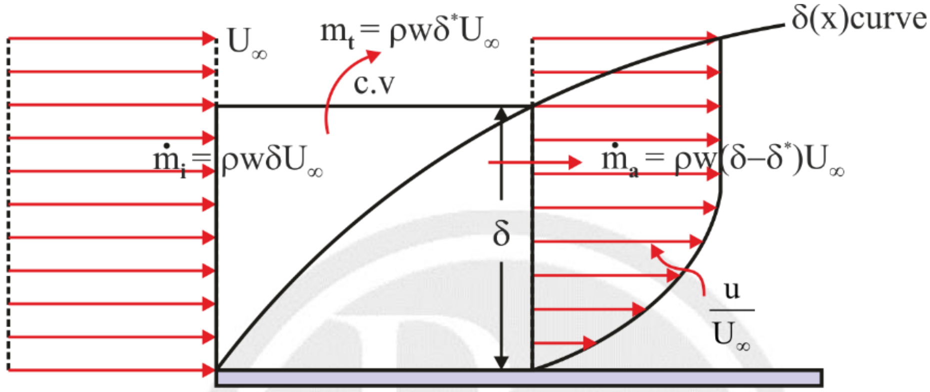

Fig.10.6

It is given by,

$$\delta^* = \int_0^{\delta}\left(1 - \frac{u}{U}\right)dy$$

Where,

$U =$ Free stream velocity

$\delta =$ Boundary layer thickness

$u =$ Velocity distribution

Physical Meaning of Displacement Thickness

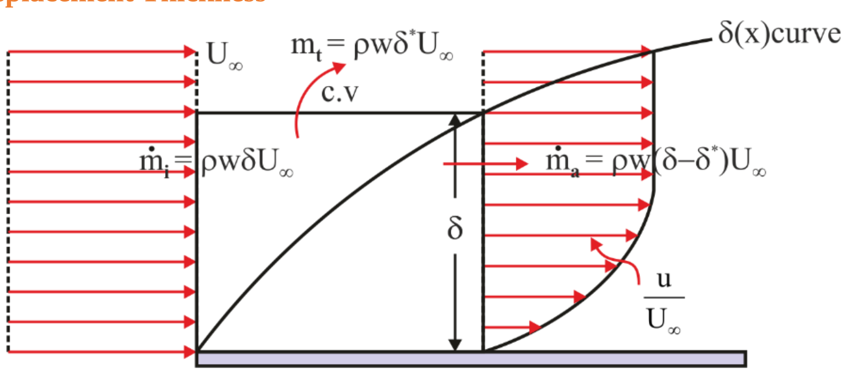

Fig. 10.7 Physical significance of displacement thickness

$$\frac{\dot{m}_0}{\dot{m}_i} = \frac{\delta - \delta^*}{\delta}$$

$$\frac{\dot{m}_t}{\dot{m}_i} = \frac{\delta^*}{\delta}$$

$\dot{m}_t = \rho w \delta^* U_\infty$

$\dot{m}_i = \rho w \delta U_\infty$

$\dot{m}_o = \rho w(\delta - \delta^*) U_\infty$

10.5.2 Momentum Thickness (θ)

- Momentum thickness represents the amount of thickness of solid plate that must be increased (imaginarily) above δ^*, so that ideal, uniform, inviscid flow has the same momentum rate as that of actual momentum rate given by actual velocity distribution.

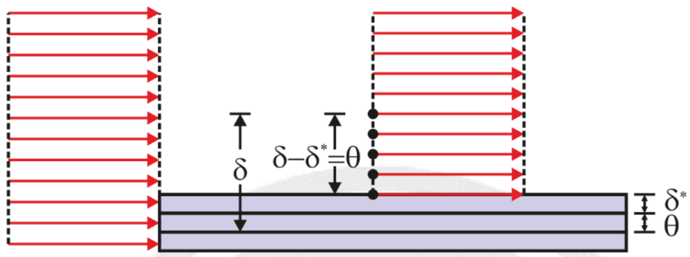

Fig. 10.8 Momentum thickness

$$\theta = \int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy$$

Physical Meaning of Momentum Thickness

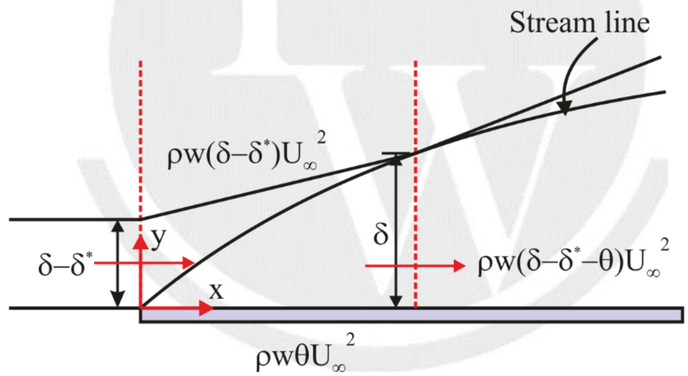

Fig. 10.9 Physical significance of momentum thickness

$$F_{d,s} = \rho \omega \theta U_\infty^2$$

10.5.3 Shape Factor

It is the ratio of displacement thickness to the momentum thickness for a given velocity distribution.

$$H = \frac{\delta^*}{\theta}$$

10.5.4 Energy Thickness (δ_e)

It is defined as the distance, measured perpendicular to the boundary of the solid body, by which the boundary should be displaced to compensate for the reduction in kinetic energy of the flowing fluid on account of boundary layer formation. It is given by,

$$\delta_e = \int_0^\delta \frac{u}{U}\left(1 - \frac{u^2}{U^2}\right) dy$$

Where,

$U =$ Free stream velocity

$\delta =$ Boundary layer thickness

$u =$ Velocity distribution

10.6 Local and Average Co-Efficient of Drag

10.6.1 Local Co-efficient of Drag (C_{fx})

It is defined as,

$$C_{fx} = \frac{\tau_{w,x}}{\frac{1}{2}\rho U^2}$$

Where,

$A =$ Area of the surface (or plate), $U =$ Free-stream velocity

$\rho =$ Mass density of fluid

$F_D =$ Total Drag force on the plate

$\tau_{w,x} =$ Shear stress at distance x from leading edge on the plate

10.6.2 Average Co-efficient of Drag (C_D)

It is defined as,

$$C_D = \frac{F_D}{\frac{1}{2}\rho A U^2}$$

Where,

$A =$ Area of the surface (or plate), $U =$ Free-stream velocity

$\rho =$ Mass density of fluid

$F_D =$ Total Drag force on the plate

10.7 Von-Karman Momentum Integral Equation

Assumptions:

Momentum thickness is the function of x only.

Pressure gradient in x-direction is zero

$$\frac{\tau_{wx}}{\rho U_\infty^2} = \frac{d\theta}{dx}$$

Where,

$\tau_{w,x} =$ shear stress on the plate at distance x from the leading edge

$\rho =$ density of fluid

$U =$ Free stream velocity

$\theta =$ Momentum thickness

10.7.1 Application of Von-Karman Momentum Integral Equation for a Given Velocity Profile

- For a given velocity profile, Von-Karman momentum integral equation is used to find
  (i) Local Wall shear stress (τ_{wx})
  (ii) Local skin friction drag coefficient (c_{fx})
  (iii) Skin friction force acting on plate ($F_{d,s}$)
  (iv) Average skin friction drag coefficient ($C_{d,s}$)

Velocity Distribution	δ	C_{fx}	C_D
$\frac{u}{U}=2\left(\frac{y}{\delta}\right)-\left(\frac{y}{\delta}\right)^2$	$5.48x/\sqrt{R_{e_x}}$	$0.73/\sqrt{R_{e_x}}$	$1.46/\sqrt{R_{e_L}}$
$\frac{u}{U}=\frac{3}{2}\left(\frac{y}{\delta}\right)-\frac{1}{2}\left(\frac{y}{\delta}\right)^3$	$4.64x/\sqrt{R_{e_x}}$	$0.646/\sqrt{R_{e_x}}$	$1.292/\sqrt{R_{e_L}}$
$\frac{u}{U}=2\left(\frac{y}{\delta}\right)-2\left(\frac{y}{\delta}\right)^3+\left(\frac{y}{\delta}\right)^4$	$5.84x/\sqrt{R_{e_x}}$	$0.68/\sqrt{R_{e_x}}$	$1.36/\sqrt{R_{e_L}}$
$\frac{u}{U}=\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)$	$4.79x/\sqrt{R_{e_x}}$	$0.655/\sqrt{R_{e_x}}$	$1.31/\sqrt{R_{e_L}}$

10.8 Prandtl's Boundary Layer Equation

Assumption: Flow is 2 − D (x − y plane), steady & incompressible flow Gravitational effects are neglected

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$$

10.9 Blasius Equation

- The classical problem considered by Blasius was a 2D, steady, incompressible fluid flow over a plate at a zero angle of incidence
- In his problem Blasius used Prandtl Boundary layer equation & Continuity equation.

  $$f'''(\eta)+\frac{1}{2}f(\eta)f''(\eta)=0$$

  Order ⇒ 3

  Non-Linear

  Ordinary Differential equation

- Boundary conditions for Blasius equation

  at $\eta=0$ $f'(\eta)=0$

at $\eta = 0$ $f(\eta) = 0$

at $\eta \to \infty$ $f'(\eta) = 1$

10.10 Boundary Layer Separation

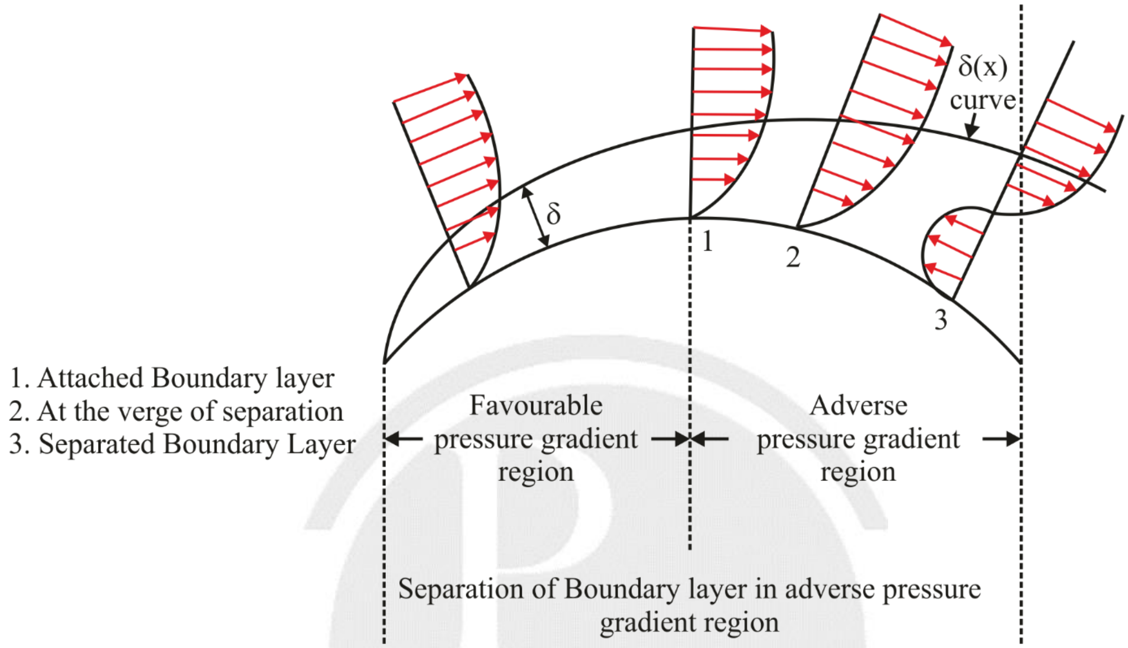

Fig. 10.10 Boundary layer separation

When a solid body is immersed in a flowing fluid, a thin layer of fluid called the boundary layer is formed adjacent to the solid body. In this thin layer of fluid, the velocity varies from zero to free-stream velocity in the direction normal to the solid body. Along the length of the solid body, the thickness of the boundary layer increases. The fluid layer adjacent to the solid surface has to do work against surface friction at the expense of its kinetic energy. This loss of the kinetic energy is recovered from the immediate fluid layer in contact with the layer adjacent to solid surface through momentum exchange process. Thus, the velocity of the layer goes on decreasing. Along the length of the solid body, at a certain point a stage may come when the boundary layer may not be able to keep sticking body. In other works, the boundary layer will be separated from the surface. This phenomenon is called the boundary layer separation. The point on the body at which the boundary layer is on the verge of separation from the surface is called point of separation.

10.10.1 Effect of Pressure on Boundary Layer Separation

1. Adverse Pressure Gradient

$$\frac{dp}{dx} > 0$$

When pressure gradient is positive $\left(\frac{dp}{dx} > 0\right)$ the velocity in the direction of flow will decreases. Thus, the velocity of the layer adjacent to the solid surface along the length of the solid surface goes on decreasing as the kinetic energy of the layer is used to overcome the frictional resistance of the surface. Hence chances of flow separation increase therefore positive pressure gradient $\left(\frac{dp}{dx} > 0\right)$ is called adverse pressure gradient.

2. Favorable Pressure Gradient

$$\frac{dp}{dx} < 0$$

When pressure gradient is negative $\left(\frac{dp}{dx} < 0\right)$ the velocity in the direction of flow will increases. Thus, the velocity of the layer adjacent to the solid surface along the length of the solid surface goes on increasing as the kinetic energy of the layer is used to overcome the frictional resistance of the surface. Hence reduces the chance of flow separation. Therefore, negative pressure gradient $\left(\frac{dp}{dx} < 0\right)$ is called favorable pressure gradient.

10.10.2 Condition for Boundary Layer Separation

The separation point S is determined from the condition $\left(\frac{\partial u}{\partial y}\right)_{y=0}$

If $\left(\frac{\partial u}{\partial y}\right)_{y=0} < 0$ Flow has separated

If $\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0$ Flow is on the verge of separation

If $\left(\frac{\partial u}{\partial y}\right)_{y=0} > 0$ Flow is attached with the surface

10.10.3 Methods for Preventing Separation

- Streamlining the body shape.
- Tripping the boundary layer from laminar to turbulent by provision of surface roughness.
- Sucking the retarded flow.
- Injecting high velocity fluid in the boundary layer.
- Providing slots near the leading edge.
- Guidance of flow in a confined passage.
- Providing a rotating cylinder near the leading edge.
- Energizing the flow by introducing optimum amount of swirl in the incoming flow.

Acceleration of Fluid in the Boundary Layer

This method of controlling separation consists of supplying additional energy to particles of fluid which are being retarded in the boundary layer. This may be achieved either by injecting the fluid into the region of boundary layer from the interior of the body with the help of some available device.

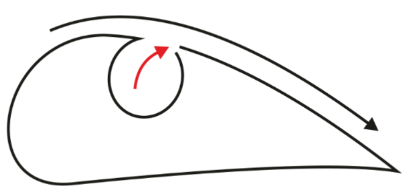

Fig. 10.11 Injecting fluid in boundary layer

By diverting a portion of fluid of the main stream from the region of high pressure to the retarded region of boundary layer through a slot provided in the body.

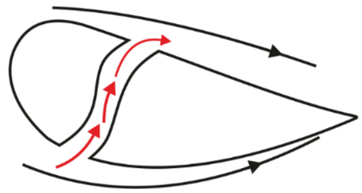

Fig. 10.12 Slotted wing

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11 DRAG AND LIFT

11.1 Drag Force

- Component of force acting on the body parallel to the relative velocity direction.

$$F_d = F_{ds} + F_{dp}$$

Here F_{ds} = Skin friction drag.

F_{dp} = Pressure drags.

$\therefore$ $F_d = \frac{1}{2}\rho A_f U_\infty^2 \times C_d$

Here, C_d = Coefficient of drag

A_f = Area

U_∞ = Free stream velocity

11.2 Analysis for Sphere

- For Creeping flow

$$Re \leq 1$$

$$U_\infty = V$$

According to stokes

$$C_d = \frac{24}{Re}$$

- Stokes law, $F_d = 3\pi\mu VD$

(i) If sphere moving downward then terminal velocity, $V = \frac{(\rho_b - \rho_f)D^2 g}{18\mu}$

(ii) If sphere moving upward then terminal velocity, $V = \frac{(\rho_f - \rho_b)D^2 g}{18\mu}$

11.3 Lift Force

Force acting on the body normal to the relative velocity direction.

$$F_l = \frac{1}{2}\rho A_p U_\infty^2 \times C_l$$

C_l = Coefficient of lift

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12 DIMENSIONAL ANALYSIS

12.1 Introduction

In the dimensional analysis of a physical phenomenon the relationship between the dependent and independent variables is studies in terms of their basic dimensions to obtain the information about the functional relationship between the dimensionless parameters that control the phenomenon.

There are several methods of reducing the number of dimensional variables into a smaller number of dimensionless parameters. Two of the commonly used methods are:

(i) Raleigh's method,

(ii) Buckingham Pi theorem method.

Advantages of dimensional analysis

1. It expresses the functional relationship between the variables in dimensionless terms.
2. It enables us in getting up the simplified dimensional form.
3. The conversion of units of quantities from the one system to another is facilitated.

Uses of dimensional analysis

1. To test the dimensional homogeneity of any equation of fluid motion
2. To derive rational formulae for a flow phenomenon
3. To plan model tests and present experimental result in a systematic manner, thus making it possible to analysis the complex fluid flow phenomenon.

12.2 Important Dimensions

Two common system of dimensioning a physical quantity are; M-L-T and F-L-T system of units, where F = force.

Quantity	Symbol	Dimensions in terms of	
		M-L-T	F-L-T
A. Geometric:			
Length (any linear measurement)	L, l	L	L
Area	A	L^2	L^2
Volume	V	L^3	L^3
Curvature	C	1/L	1/L

B	**Kinematic:**			
	Discharge per unit width	q	L^2/T	L^2/T
	Kinematic viscosity (μ/ρ)	μ	L^2/T	L^2/T
	Circulation	Γ	L^2/T	L^2/T
	Time	T, t	T	T
	Velocity (linear)	V, v	L/T	L/T
	Angular velocity	ω	1/T	1/T
	Frequency	n	1/T	1/T
	Acceleration (linear)	a	L/T^2	L/T^2
	Angular acceleration	α	$1/T^2$	$1/T^2$
	Discharge	Q	L^3/T	L^3/T
C.	**Dynamic:**			
	Mass	M, m	M	FT^2/L
	Force	F	ML/T^2	F
	Mass density	ρ	M/L^3	FT^2/L^4
	Specific weight	ω, γ	M/L^2T^2	F/L^3
	Specific gravity	s	$M^oL^oT^o$	$F^oL^oT^o$
	Pressure intensity	p	M/LT^2	F/L^2
	Shear stress	τ	M/LT^2	F/L^2
	Dynamic viscosity	μ	M/LT	FTL^2
	Surface tension	σ	M/T^2	F/L
	D. General:			
	Moment of a force	$F \times l$	ML^2/T^2	FL
	Moment of Inertia of area	I	L^4	L^4
	Moment of Inertia of mass	Ml^2	ML^2	FLT^2

12.3 Raleigh's Method

If A_1 is dependent variable and $A_2, A_3, \ldots An$ are independent variables in a phenomenon then A_1 is expressed as:

$$A_1 = k A_2^a A_3^b \ldots\ldots \qquad \ldots(i)$$

Where k is a dimensionless constant.

The dimensions of each of the quantities $A_2, A_3, \ldots An$ are written and the sum of exponents of each of M, L, and T are equated on both sides. Solution of the equations on simplification gives the value of $a, b, c \ldots$. Putting these values in equation (i) yields dimensionless groups controlling the phenomenon. While this method is simple for a small number of parameters, it becomes rather cumbersome when a large number of parameters are involved.

12.4 Buckingham's Pi Theorem

The Buckingham Pi theorem states that if there are m primary dimensions involved in the n variables controlling a physical phenomenon, then the phenomenon can be described by (n-m) independent dimensionless groups (known as πs). The word Pi here refers to a product of variables and the Greek letter π is used to indicate these products, for example π_1, π_2…

In the application of this method, m numbers of repeating variables are selected and dimensionless groups obtained by each one of the remaining variables one at a time. Raleigh's method is used to solve this part of the operation. This method is also known as the method of repeating variables.

Care is needed in selecting the repeating variables.

- Number of repeated variables must be equal to number of fundamental dimensions
- Repeated variables must be selected from the independent variables only
- Repeated variables must contain all the fundamental dimensions.
- Repeated variables must be selected such that they are not forming dimensionless terms all together.
- Most fundamental quantity must be given preferences.

12.5 MODEL ANALYSIS

Model analysis is the study of the structure or machine by constructing a small-scale replica of the machine or structure known as model and performing the various tests so as to detect and remove various defects in the design of the actual structure called prototype.

For predicting the performance of the hydra

.0ulic structures or hydraulic machine, before actually constructing or manufacturing, models of the structures or machines are made and test are performed on them to obtain the desired information.

Mostly the models are much smaller than the corresponding prototypes, but in some cases the model may be larger than the prototype.

Advantages of Model Analysis

1. The model tests are quite economical and convenient because the design, construction and operation of the model may be altered several times without incurring much expenditure.
2. The performance of the hydraulic structure or hydraulic machines can be easily predicted.
3. Model tests are used to detect and rectify the defects of an existing structure which is not functioning property.
4. With the help of dimensional analysis, a relationship between the variables influencing a flow problem in terms of dimensionless parameters is obtained. This relationship helps in conducting tests on the model.

Application of the Model Analysis

1. Design of turbine, pumps and compressors.
2. Design of harbors, ships and submarines.
3. Design of dams, spillways, wires, canals etc.

12.5.1 Similitude Principle

In hydraulic and aeronautical engineering valuable results are obtained at a relatively small cost by performing tests on small scale models of full-size systems (prototypes). Similarity laws help us interpret the results of model studies. Similitude is the relation between model and a prototype and is classified into three kinds as follows:

(i) Geometric similarity

(ii) Kinematic similarity

(iii) Dynamic similarity

Geometric Similarity

If the ratios of corresponding length in a model and the prototype are the same, the model is said to be in geometrically similar model.

In such models

If $$\frac{L_m}{L_p} = L_r$$

Then, $$\frac{A_m}{A_p} = \frac{(L_m)^2}{(L_p)^2} = L_r^2,$$

$$\frac{\cancel{V}_m}{\cancel{V}_p} = \frac{(L_m)^3}{(L_p)^3} = L_r^3$$

Where,

L_m = Length of Model

A_m = Area of model

$\cancel{V}_m$ = Volume of Model

Similarly, L_p, A_p & $\cancel{V}_p$ are corresponding parameter for prototype.

Kinematic Similarity

Kinematic similarity means geometric similarity and in addition it also signifies that the ratio of velocities at all corresponding points in the flow for model & prototype is the same.

If $$\frac{L_m}{L_p} = L_r$$

$$\frac{V_m}{V_p} = V_r$$

Then

(i) Time ratio $= \dfrac{t_m}{t_p} = t_r = \dfrac{L_r}{V_r}$

(ii) Acceleration ratio $= \dfrac{a_m}{a_p} = a_r = \dfrac{V_r^2}{L_r} = \dfrac{L_r}{T_r^2}$

(iii) Discharge ratio $= \dfrac{Q_m}{Q_p} = Q_r = \dfrac{L_r^3}{T_r}$

Dynamic Similarity

Two systems are dynamically similar, if geometric and kinematic similarities exist and further the ratios of all corresponding forces in the two systems are same.

If forces due to

Gravity = F_G

Viscosity = F_v

Elasticity = F_E

Surface tension = F_T

Inertia = F_1

And suffixes m and p stand for model and prototype respectively, strict dynamic similarity means

$$\frac{F_{Gm}}{F_{Gp}} = \frac{F_{vm}}{F_{vp}} = \frac{F_{Em}}{F_{Ep}} = \frac{F_{Tm}}{F_{Tp}} = \frac{F_{1m}}{F_{1p}} = \text{constant}$$

From the above the following relationships can be derived.

$$\frac{(\text{Inertia force})_m}{(\text{Viscous force})_m} = \frac{(\text{Inertia force})_p}{(\text{Viscous force})_p} = \text{Constant 1}$$

$$\frac{(\text{Inertia force})_m}{(\text{Gravity force})_m} = \frac{(\text{Inertia force})_p}{(\text{Gravity force})_p} = \text{Constant 2}$$

And so on for all the forces

12.6-Dimensional Formula of Various Forces

- Inertia force $(F_i) = \rho V^2 L^2$
- Viscous force $(F_v) = \mu VL$
- Gravity force $(F_g) = \rho L^3 g$
- Pressure force $(F_p) = pL^2$
- Surface tension force $(F_{st}) = \sigma L$
- Elastic force $(F_e) = KL^2$

  where K is bulk modulus of Elasticity

12.6.1 Dimensionless Numbers

Number	Equation	Significance
Reynolds No	$\frac{\rho VL}{\mu}$	Flow in closed conduit pipe
Froude No.	$\frac{V}{\sqrt{Lg}}$	Where a free surface is present, structure eg. Weirs, spillway, channels, etc. where gravity force is predominant
Euler's No.	$\frac{\rho V^2}{p}$	In cavitation studies
Mach No.	$\frac{V}{C}$	Where fluid compressibility is important.
Weber No.	$\frac{\rho V^2 L}{\sigma}$	In capillary studies.

12.7 Model Law

12.7.1 Reynold's Model Law

For the flows where in Viscous Force is predominant in addition to inertia force, similarity can be established by equating Reynolds Number of models $(R_e)_m$ and prototype $(R_e)_p$. This is known as Reynold's Model Law.

$$(R_e)_m = (R_e)_p\ ,$$

$$\frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_p V_p L_p}{\mu_p}$$

Applications of Reynolds Model Law

- (i) Air planes, (ii) Torpedo's
- Flow through pipes, flow past plates, Fluid drag and lift on body shapes (cars, trains, parachutes… etc.).
- Settling of particles and creeping flow
- Venturi meters, Orifice meters, etc.
- Fans, blowers, propellers, pumps and turbines.

12.7.2 Froude's Model Law

When the force of gravity is predominant in addition to inertia force then similarity can be established by equating Froude's number of model & prototype. This is known as Froude's model law.

$$(F_e)_m = (F_e)_p$$

$$\frac{V_m}{\sqrt{L_m g_m}} = \frac{V_p}{\sqrt{L_p g_p}}$$

Applications of Froude's Model Law

- (i) Motion of ships, boats, (ii) Break waters and harbors
- Flow in canals, streams and rivers
- Spillways, stilling basins, hydraulic/jumps, weirs and notches
- Wave and water flow forces in bridge piers, off-shore structures, jetties and piers

12.7.3 Euler's Model Law

When pressure force is predominant in addition to inertia force, similarity can be established by equating Euler number of model and prototype. This is called Euler's model law.

$$(E_u)_m = (E_u)_p$$

$$\frac{V_m}{\sqrt{P_m / \rho_m}} = \frac{V_p}{\sqrt{P_p / \rho_p}}$$

Applications of Euler's Model Law

- Enclosed fluid system where the turbulence is fully developed so that viscous forces are negligible and also the forces of gravity and surface tension are entirely absent.
- Where the phenomenon of cavitation occurs.

12.7.4 Weber Model Law

If surface tension forces are predominant with inertia force, similarity can be established by equating Weber number of model and prototype

$$(W_e)_m = (W_e)_p$$

$$\frac{V_m}{\sqrt{\sigma_m / \rho_m L_m}} = \frac{V_p}{\sqrt{\sigma_p / \rho_p L_p}}$$

Applications of Weber Model Law

- Capillary rise in narrow passages.
- Capillary movement of water in soil.
- Capillary waves in channels.
- Flow over weirs for small heads.

12.7.5 Mach Model Law

In places where elastic forces are significant in addition to inertia, similarity can be achieved by equating Mach numbers for both the system. This is known as Mach model law.

$$(M)_m = (M)_p$$

$$\frac{V_m}{\sqrt{K_m / \rho_m}} = \frac{V_p}{\sqrt{K_p / \rho_p}}$$

Applications of Mach Model Law

- Aerodynamic testing.
- Under water testing of torpedoes.
- Water-hammer problems.

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13 Basic Compressible Flow

13.1 Introduction

13.1.1 Mach Number (Ma)

$$\text{Mach Number (Ma)} = \frac{\text{Velocity of fluid flow (V)}}{\text{Speed of sound in fluid (c)}}$$

$\text{Ma} \leq 0.3 \Rightarrow$ Incompressible Flow

$\text{Ma} > 0.3 \Rightarrow$ Compressible Flow

Here $c = \sqrt{\frac{dP}{d\rho}}$

13.1.2 For Isentropic Flow

Speed of sound $c = \sqrt{\frac{k_s}{\rho}}$

Here k_s = Isentropic Bulk Modulus

$$k_s = \rho \left.\frac{\partial P}{\partial \rho}\right|_s$$

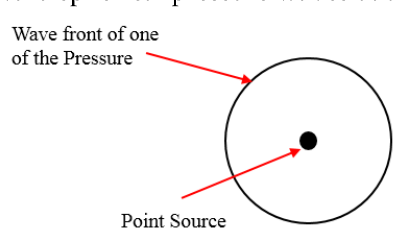
13.2 For Isentropic Flow of Perfect Gas

$$c = \sqrt{\gamma RT}$$

Here T = Absolute temperature in Kelvin.

13.2.1 Point Source

- Point source will emit the radially outward spherical pressure waves at the speed of sound.



- In Stationary point source in Stationary Fluid, a stationary observer will hear the same sound frequency coming from the source.
- In Moving point source ($V < c$) in stationary fluid ($Ma < 1$), a stationary observer will hear the different sound frequency coming from the source.
- In Moving point source ($V = c$) in stationary fluid ($Ma = 1$), formation of Mach wave (Tangent Plane)
- In moving point source ($V > c$) in stationary fluid ($Ma > 1$)
- $\sin\alpha = \frac{1}{Ma}$, Formation of Mach wave (Mach Cone).

13.3 Stagnation Temperature

$$T_o = T + \frac{V^2}{2c_p} \qquad \text{(Adiabatically perfect gas)}$$

$$\frac{T_o}{T} = 1 + \frac{(\gamma-1)}{2}Ma^2 \qquad \text{(Isentropically perfect gas)}$$

13.3.1 Stagnation Pressure

$$\frac{P_o}{P} = \left[1 + \frac{(\gamma-1)}{2}Ma^2\right]^{\frac{\gamma}{(\gamma-1)}} \qquad \text{(Isentropically perfect gas)}$$

For Incompressible fluid flow

$$\frac{P_o}{P} = 1 + \frac{\gamma}{2}Ma^2 \qquad \text{(Incompressible fluid flow)}$$

13.3.2 Stagnation Density

$$\frac{\rho_o}{\rho} = \left[1 + \left(\frac{\gamma-1}{2}\right)Ma^2\right]^{\frac{1}{(\gamma-1)}} \qquad \text{(Isentropically perfect gas)}$$

- Stagnation properties are defined at each & every point of fluid flow.

Note:

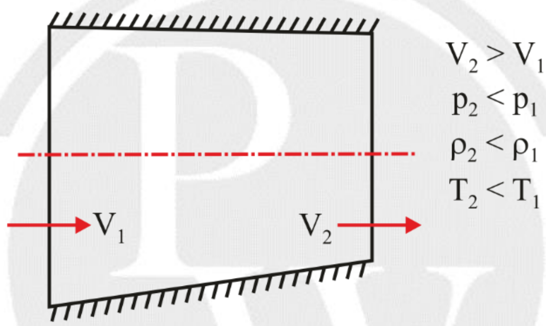
- Stagnation properties (h_o, T_o, P_o, ρ_o) are the combination of thermodynamic properties, hence stagnation properties are defined at each of every point of fluid flow.
- If we want to achieve particular value of stagnation properties then we need to bring the fluid at rest under the discussed conditions.

13.4 Effect of Area Variation on Flow Properties

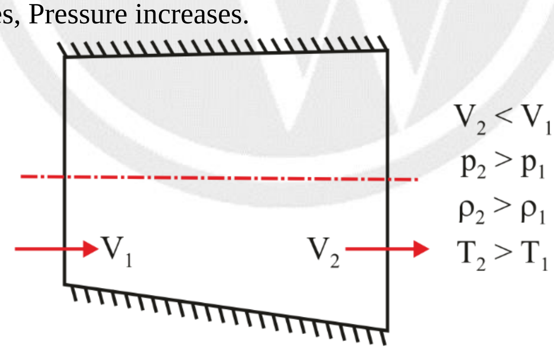
$$\frac{dA}{A} = \frac{dP}{\rho V^2}(1 - Ma^2)$$

$$\frac{dA}{A} = -\frac{dV}{V}(1 - Ma^2)$$

- At throat, Mach number will be either maximum or minimum of converging duct.
- At throat, it is not compulsory to have Ma = 1 of converging duct.
- If in the fluid flow at any section Mach is 1, that has to be throat.
- At $Ma = 1$(Sonic condition), $\frac{dA}{A} = 0 \Rightarrow$ Area is minimum at throat, so V is maximum.

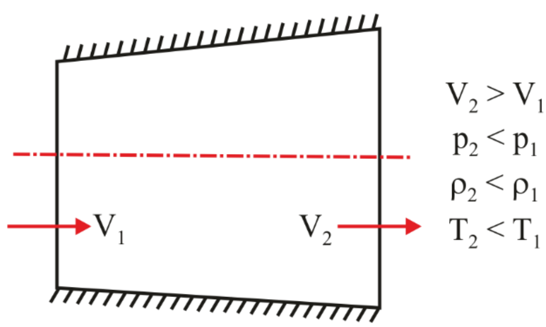


Diffuser Effect Velocity decreases, Pressure increases.

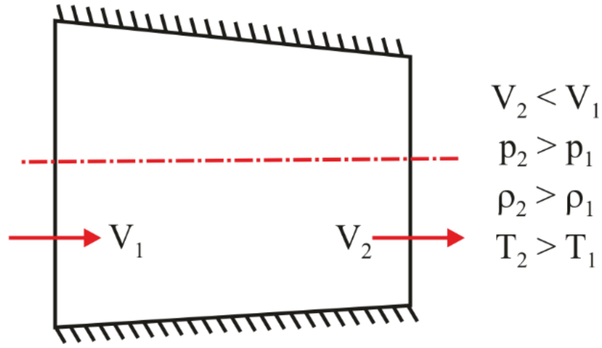


- Super Sonic Flow → $Ma > 1$

Nozzle Effect Velocity decreases, Pressure increases.

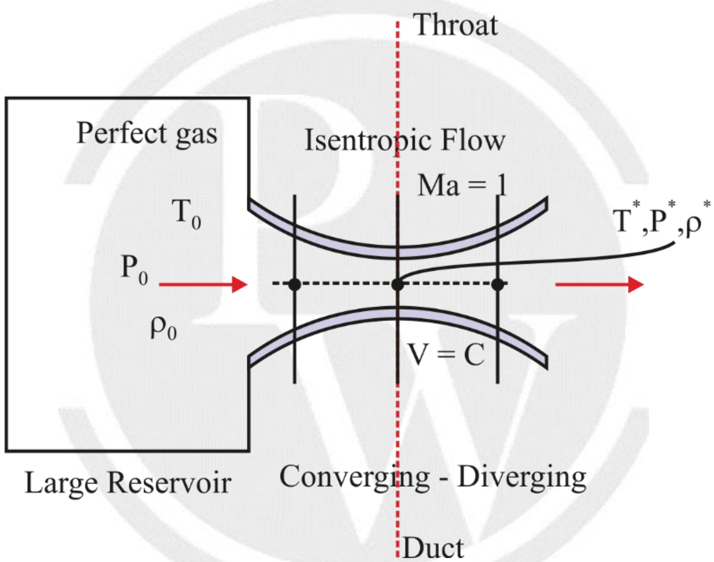


Diffuser Effect Velocity increases, Pressure decreases



13.4.1 Sonic Properties

- Ma = 1, V = c

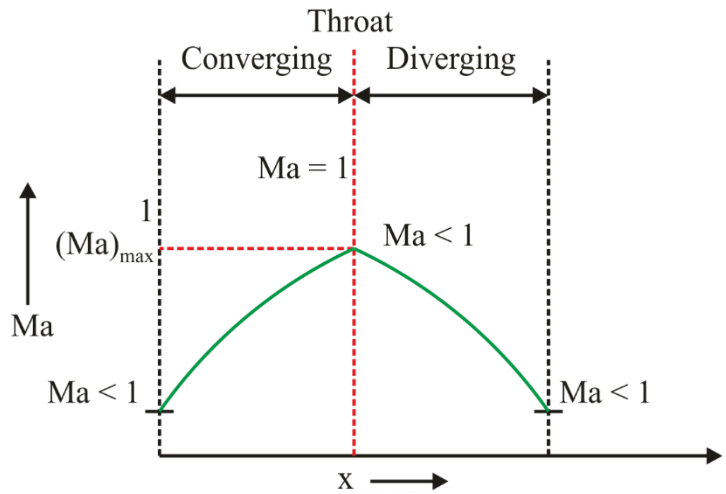


$$\frac{T_0}{T^*} = \frac{\gamma + 1}{2}$$

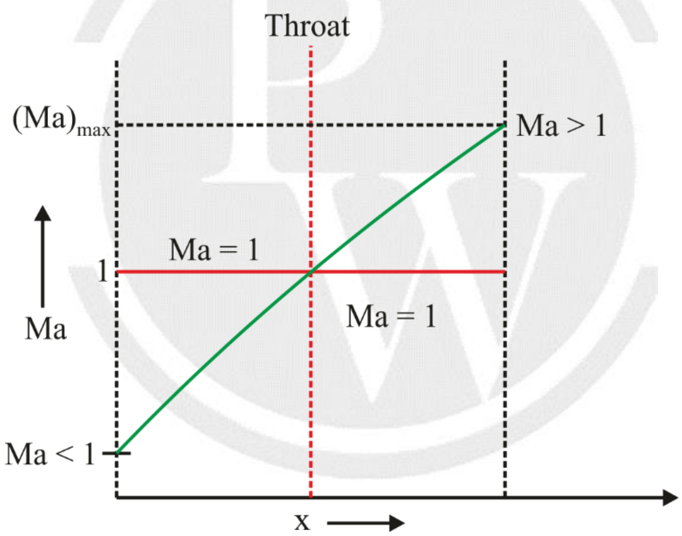
$$\frac{\rho_o}{\rho^*} = \left[\frac{\gamma + 1}{2}\right]^{\frac{1}{\gamma-1}}$$

$$\frac{P_o}{P^*} = \left[\frac{\gamma + 1}{2}\right]^{\frac{\gamma}{\gamma-1}}$$

13.4.2 Subsonic Flow at Both Inlet & Outlet

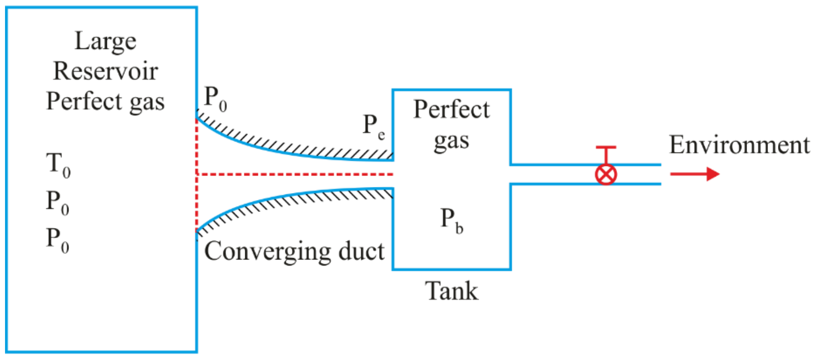


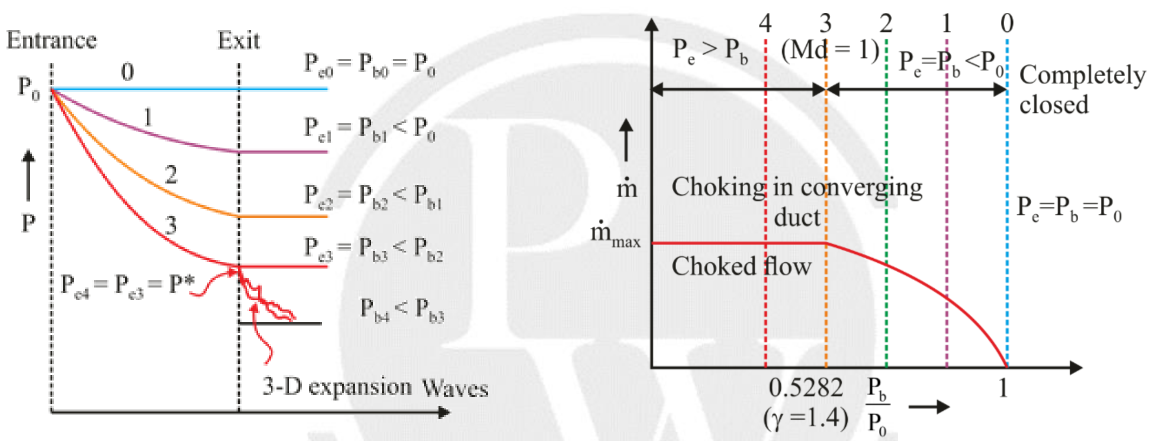
13.4.3 Subsonic Flow at the Inlet & Supersonic Flow at Outlet



Maximum mass flow rate will occur at throat (Ma = 1)

13.5 Pressure distribution in a Converging duct





For perfect gas, $\gamma = 1.4$

$$\frac{P_0}{P^*} = \left[\frac{\gamma+1}{2}\right]^{\frac{\gamma}{\gamma-1}}$$

$$\frac{P_0}{P^*} = 1.8929$$

$$\frac{P^*}{P_0} = 0.5282$$

❑❑❑